2π is not zero (but 4π is)

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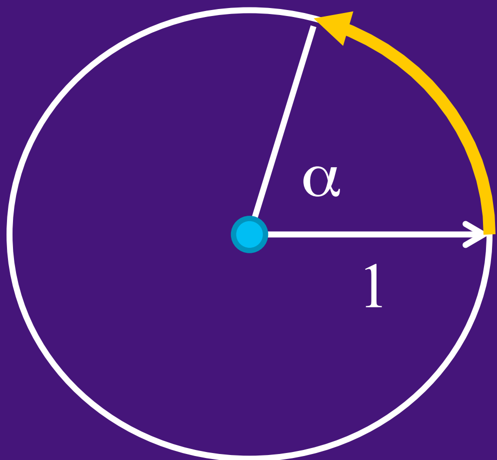
2π?



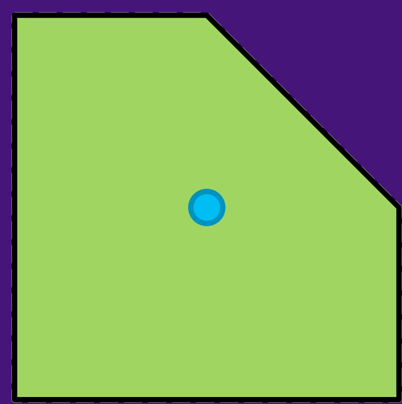
How mathematicians think about angle: the unit circle

Angle measure in "radians": length of arc on unit circle

Circumference = $2\pi \times$ radius

Degrees	Radians	What it means
90°	$\pi/2$	A quarter-turn
180°	π	Halfway around
360°	2π	One complete turn
720°	4π	Two complete turns

Of course 2π = 0!



If we rotate a geometrical shape by 2π radians (360°), there is no net change to the shape.

Also true in 3-D space:

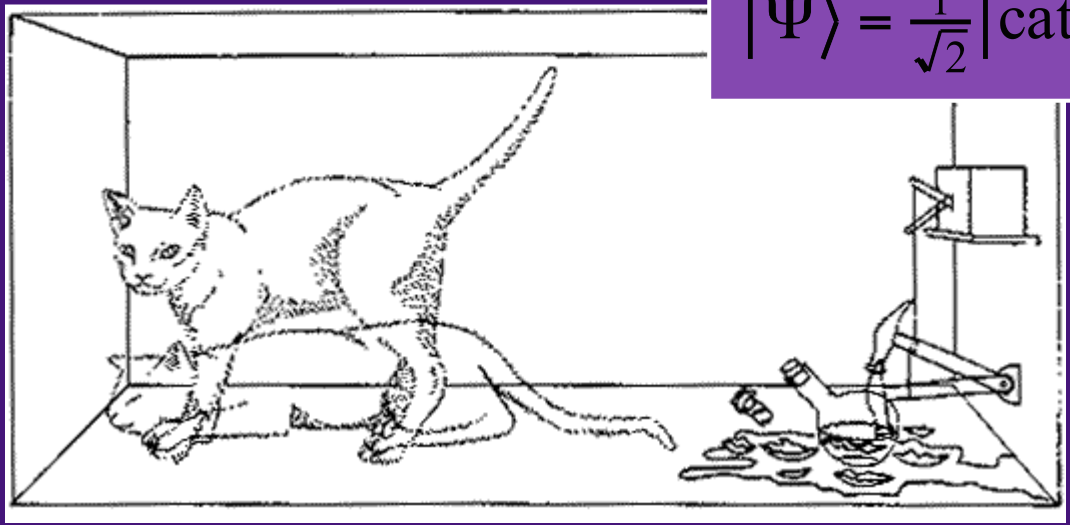
- any shape
- any axis of rotation
- Rot(2π) = Rot(0) = 1

A quantum puzzle

Quantum states

- Physical situation described by a mathematical object: the quantum state $|\Psi\rangle$
- Some quantum states describe familiar situations: $|\text{cat alive}\rangle$, $|\text{cat dead}\rangle$
- Objects can also be in a "superposition" state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|\text{cat alive}\rangle + \frac{1}{\sqrt{2}}|\text{cat dead}\rangle$$

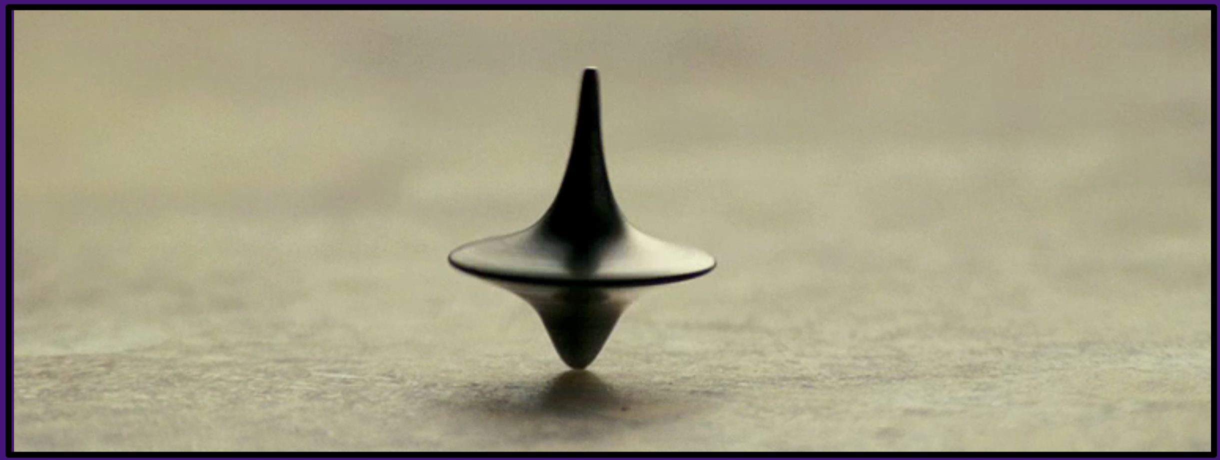


Observe the cat:

$$P(\text{alive}) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$P(\text{dead}) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

Quantum spin



- Particles have an "internal" angular momentum called spin.
- Total spin can be 0, 1/2, 1, 3/2, etc. (in units of $\hbar$).
- Electrons, protons and neutrons have spin 1/2.
- Spin is related to magnetic properties -- we can affect and measure spin using magnetic fields.

A curious fact about rotation

- Spin-1/2 particle
- Start with any spin state $|\phi\rangle$
- Rotate spin by 2π (360°): $|\phi\rangle \rightarrow \mathrm{Rot}(2\pi)|\phi\rangle = -|\phi\rangle$

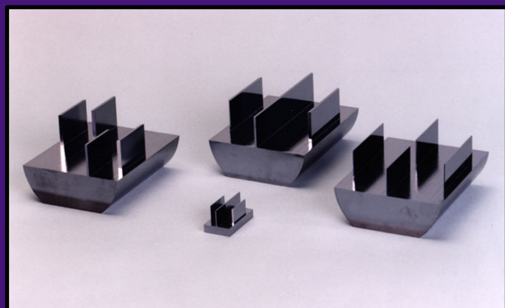
weird minus sign

- In effect, Rot(2π) = -1, not +1
- To return to the initial quantum state, we must rotate the spin by 4π (720°). Rot(4π) = +1!

Does this fact have any observable consequences?

(*Some books say "no" -- but they're wrong*.)

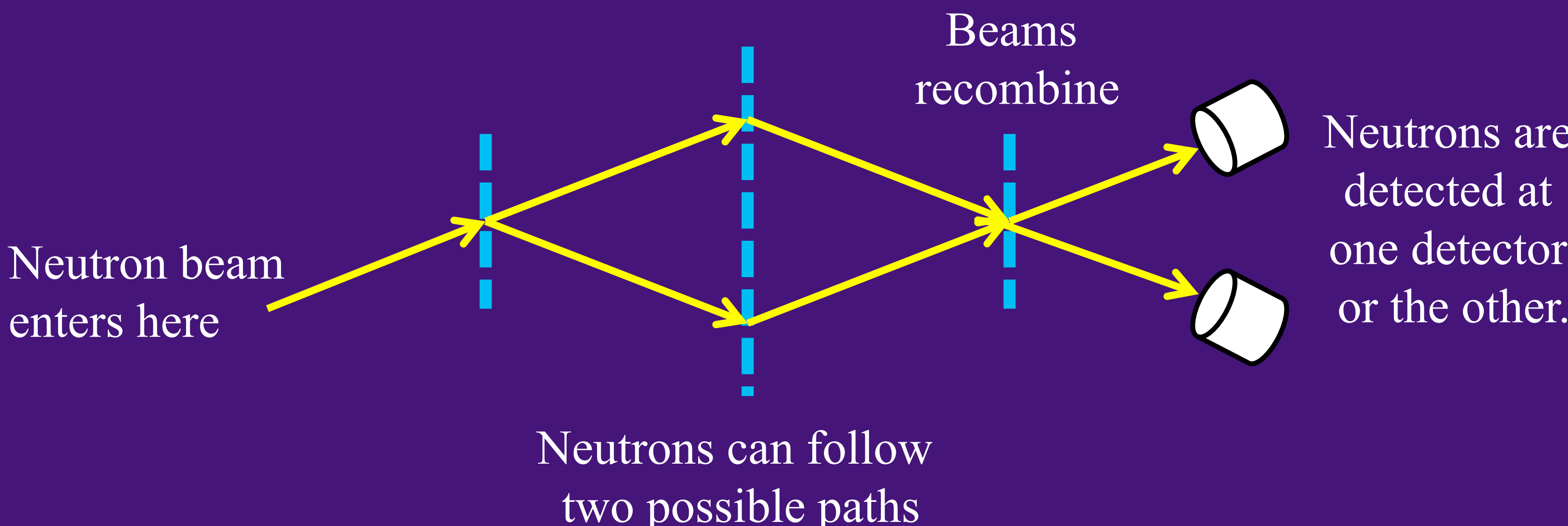
Neutron interferometry



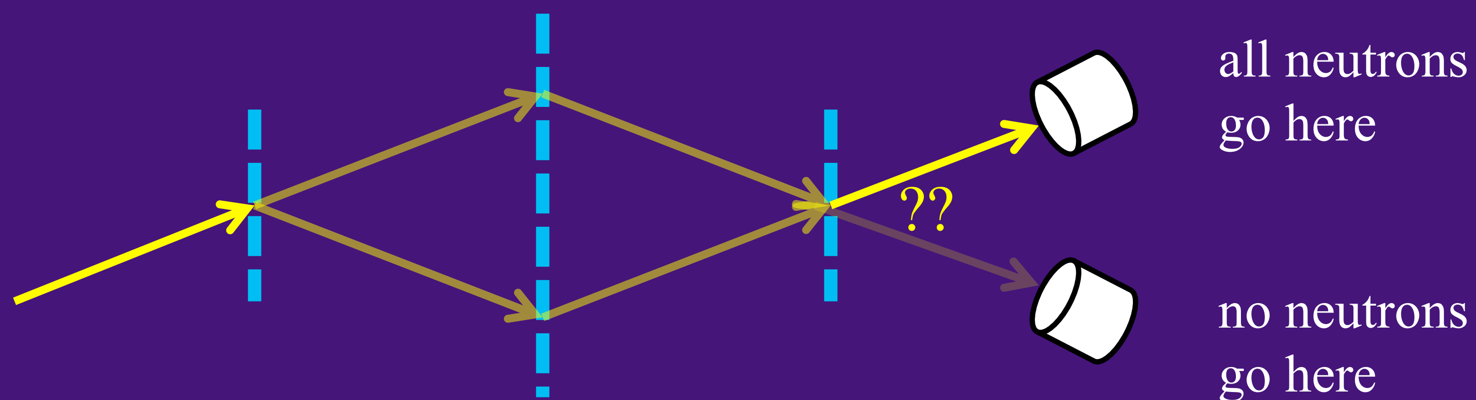
Si single-crystal neutron interferometers at NIST

Quantum physics: Neutrons travel through space as waves.

We can arrange for these waves to interfere with each other.



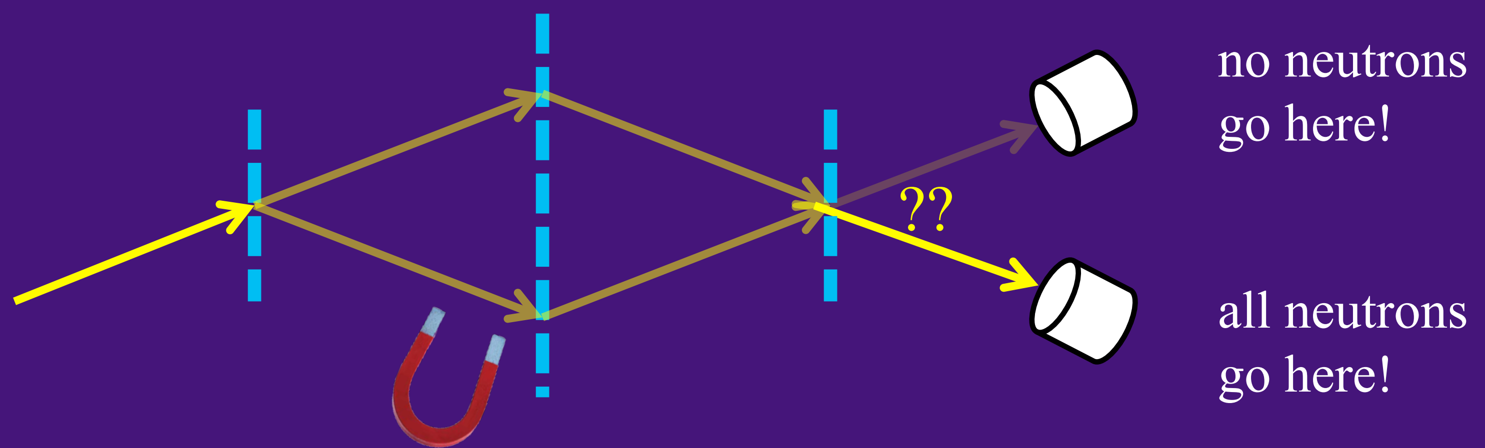
Neutron interferometry



Constructive interference: Waves add up to a more intense wave

Destructive interference: Waves cancel out to zero

Neutron interferometry



Use magnetic field to rotate neutron spins in lower beam by 2π

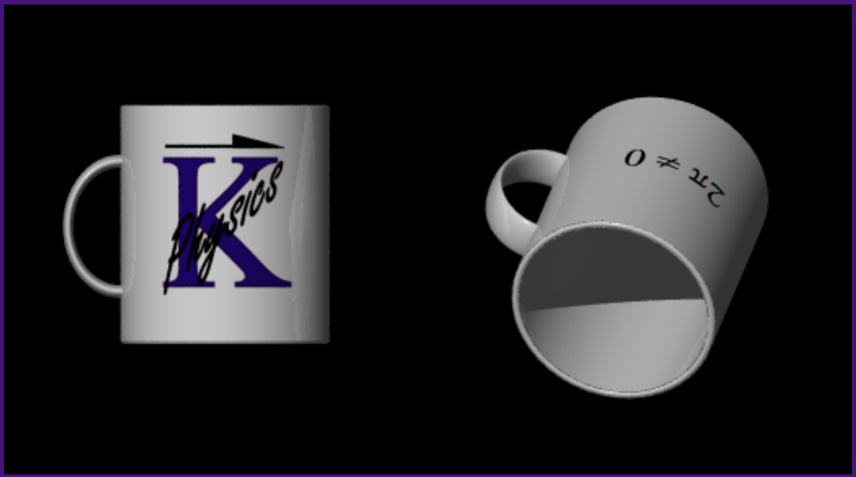
Note: If we rotate by 4π, we get back the original pattern.

Relative minus sign changes constructive to destructive interference and *vice versa* -- can be (and is) observed!

Rot(2π) is not the same as Rot(0), but Rot(4π) is!

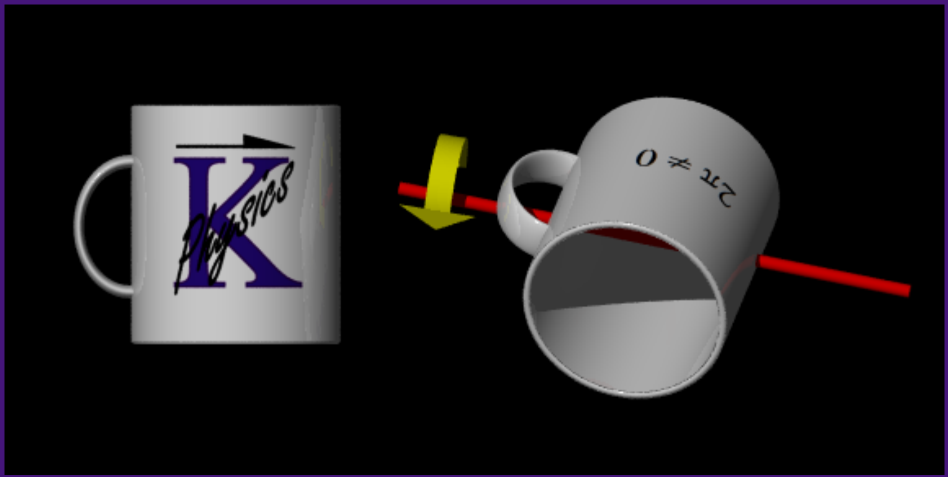
A visit to O-space

Orientation



- When we rotate an object, we change its orientation in space.
- Think of this as a kind of "motion" in "orientation space" (O-space).
- Each possible orientation of the object is a "point" in O-space.
- What does O-space look like for 3-D objects?

Euler's rotation theorem



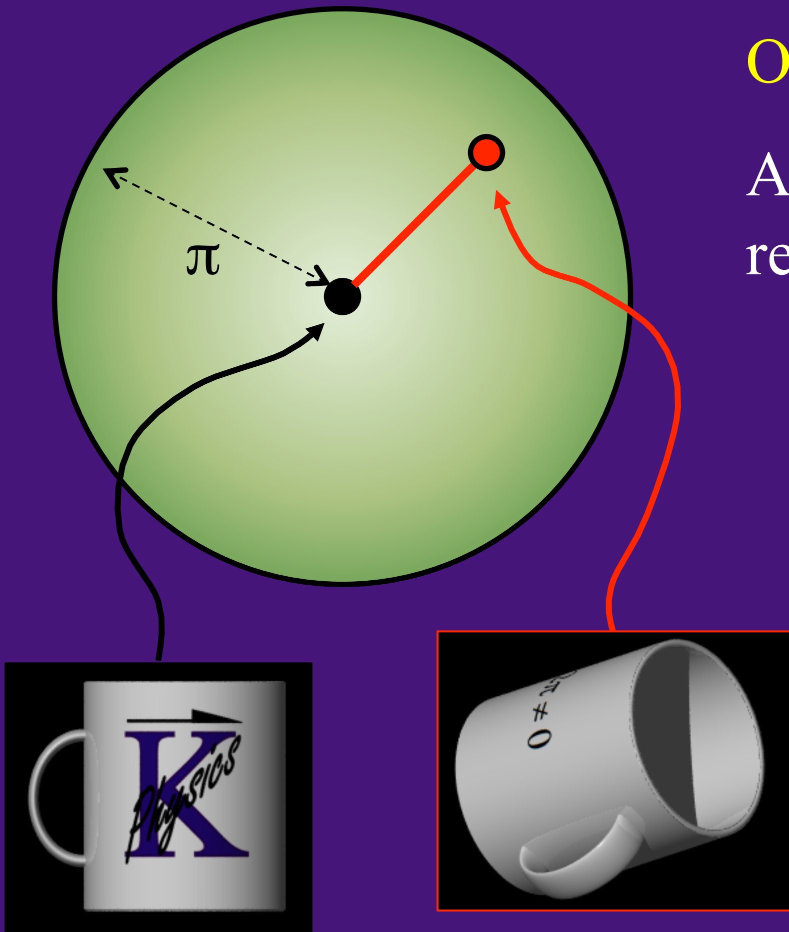
Leonhard Euler
(1707-1783)

Any point in O-space can be reached by

- choosing an axis in space, and
- rotating about that axis by some angle.

Rotation direction is given by "right-hand rule".
Rotation angle is between 0 and π.

A map of O-space



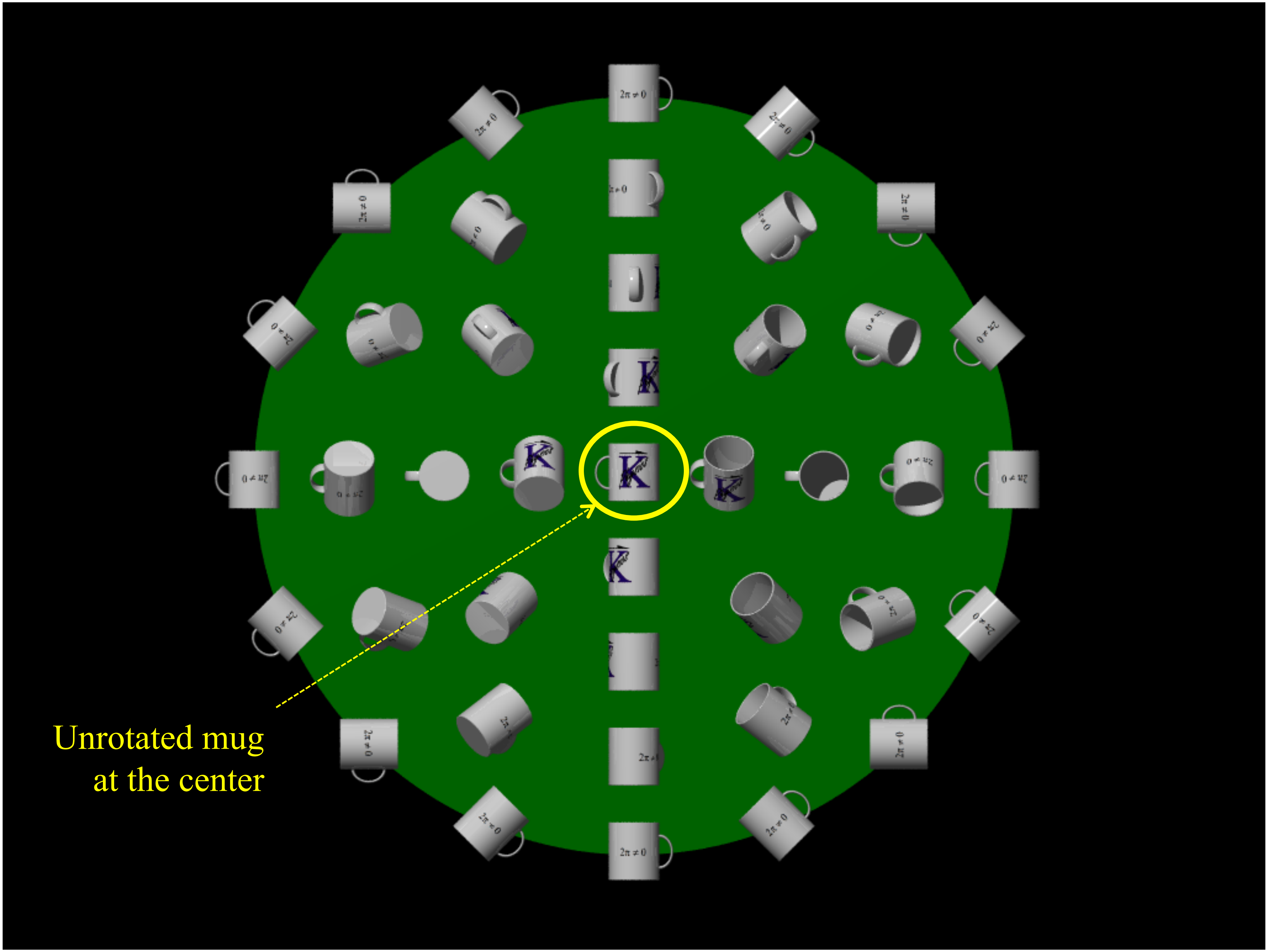
O-space is a sphere of radius π.

Antipodal points on the sphere are really the same point in O-space.

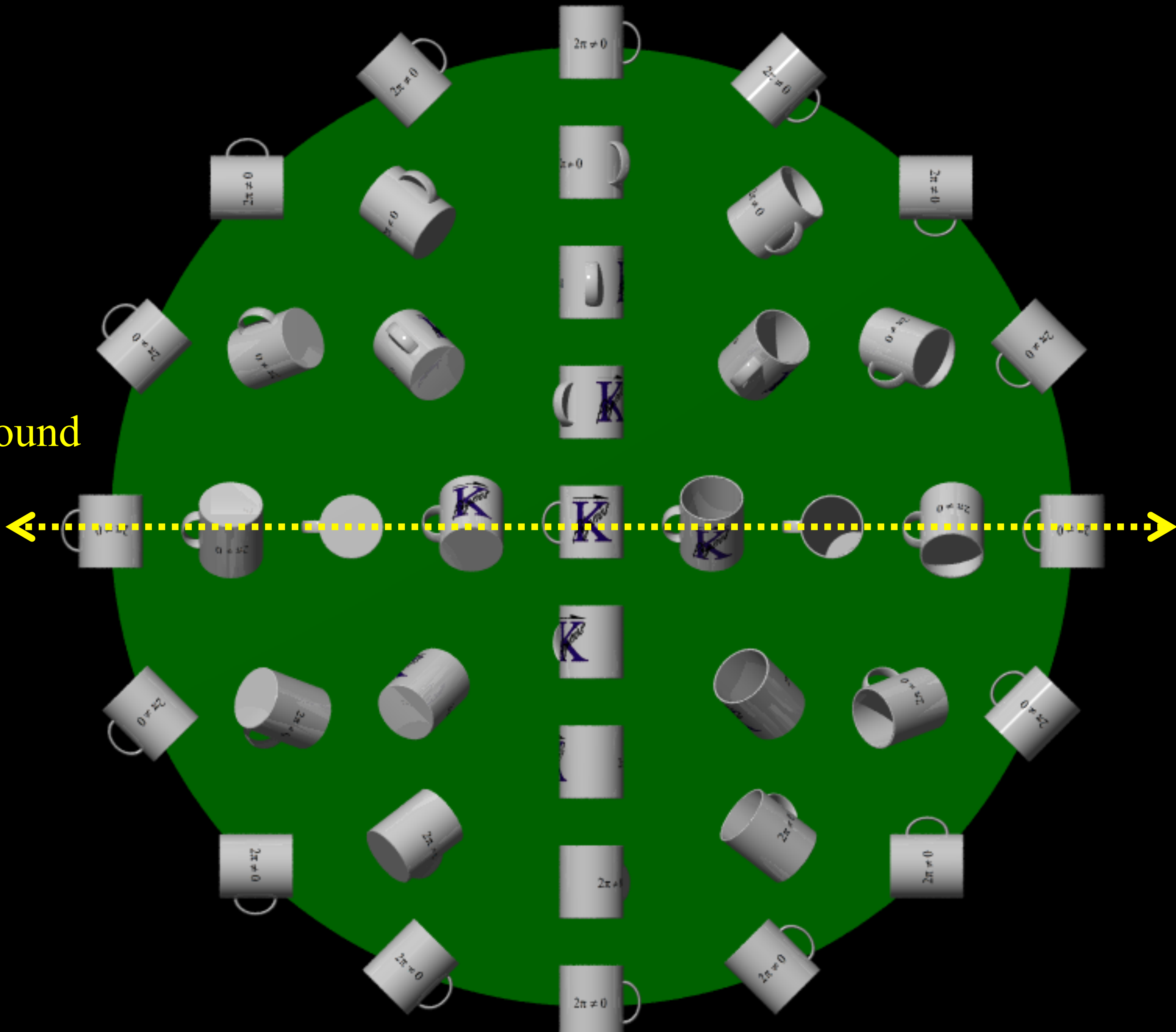
Direction indicates axis.

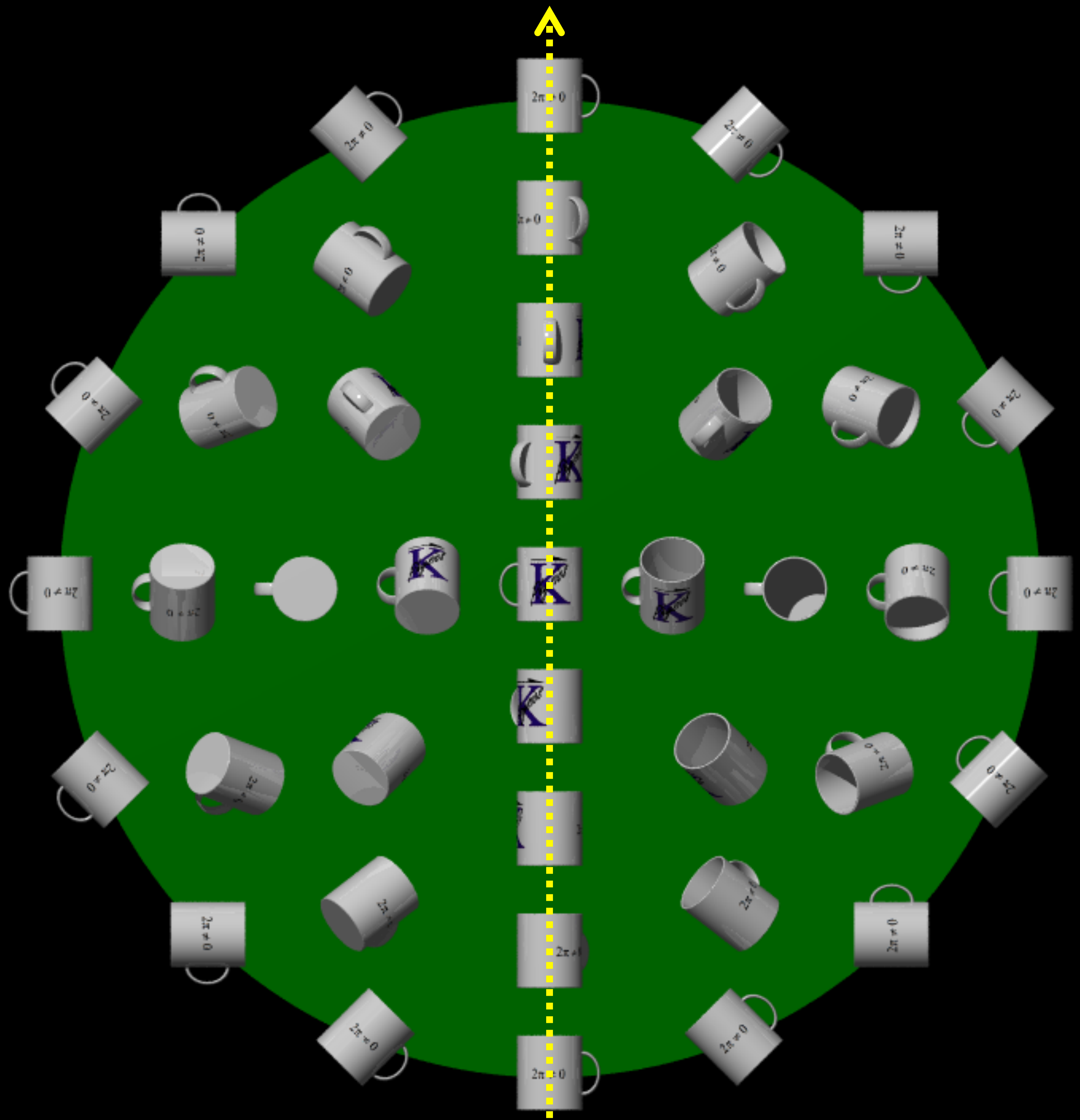
Distance from center indicates angle (0 to π).

It's hard to think in 3-D -- let's consider the "xy-slice" across O-space.

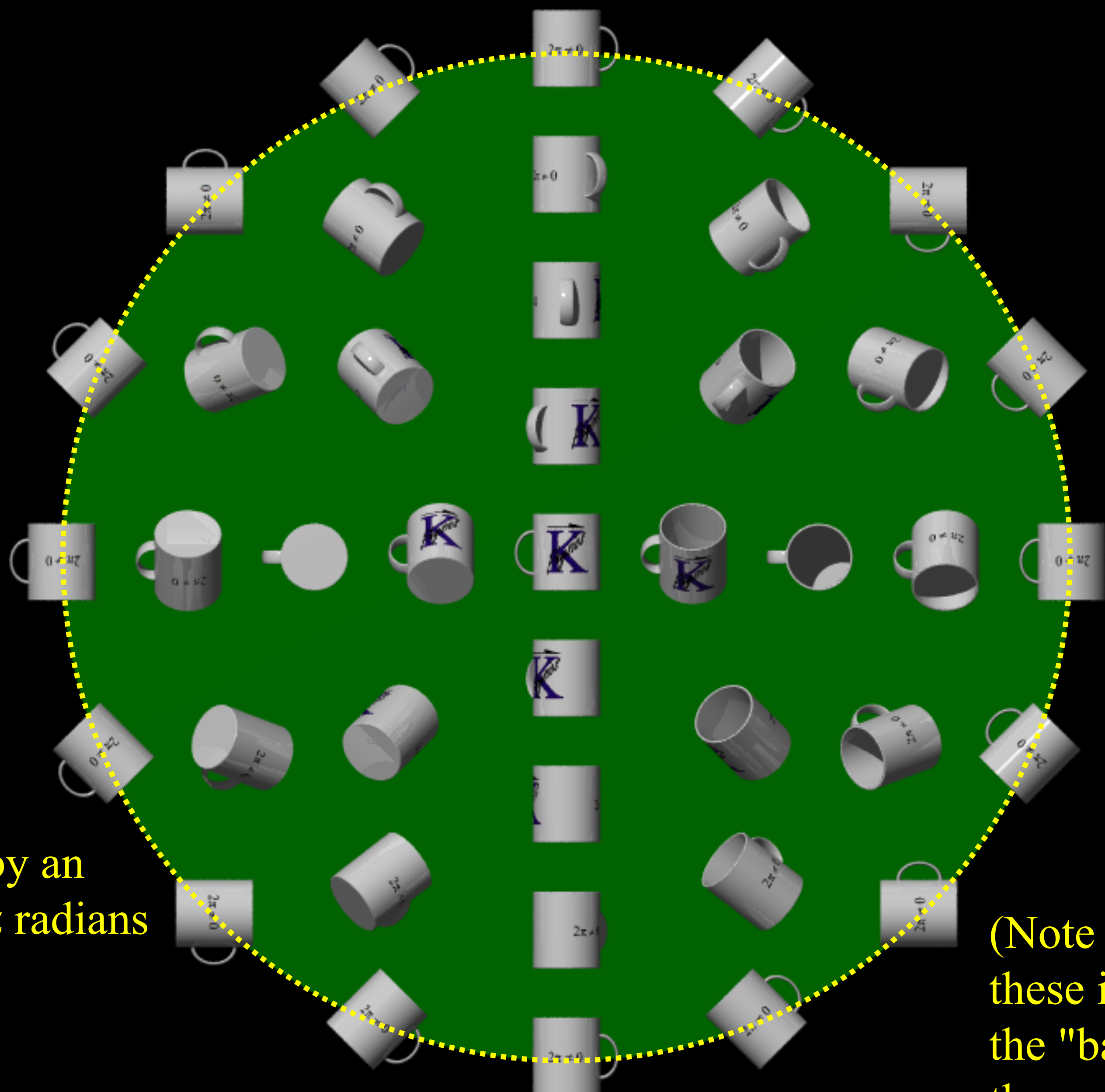

Unrotated mug
at the center

Rotated around the x-axis.



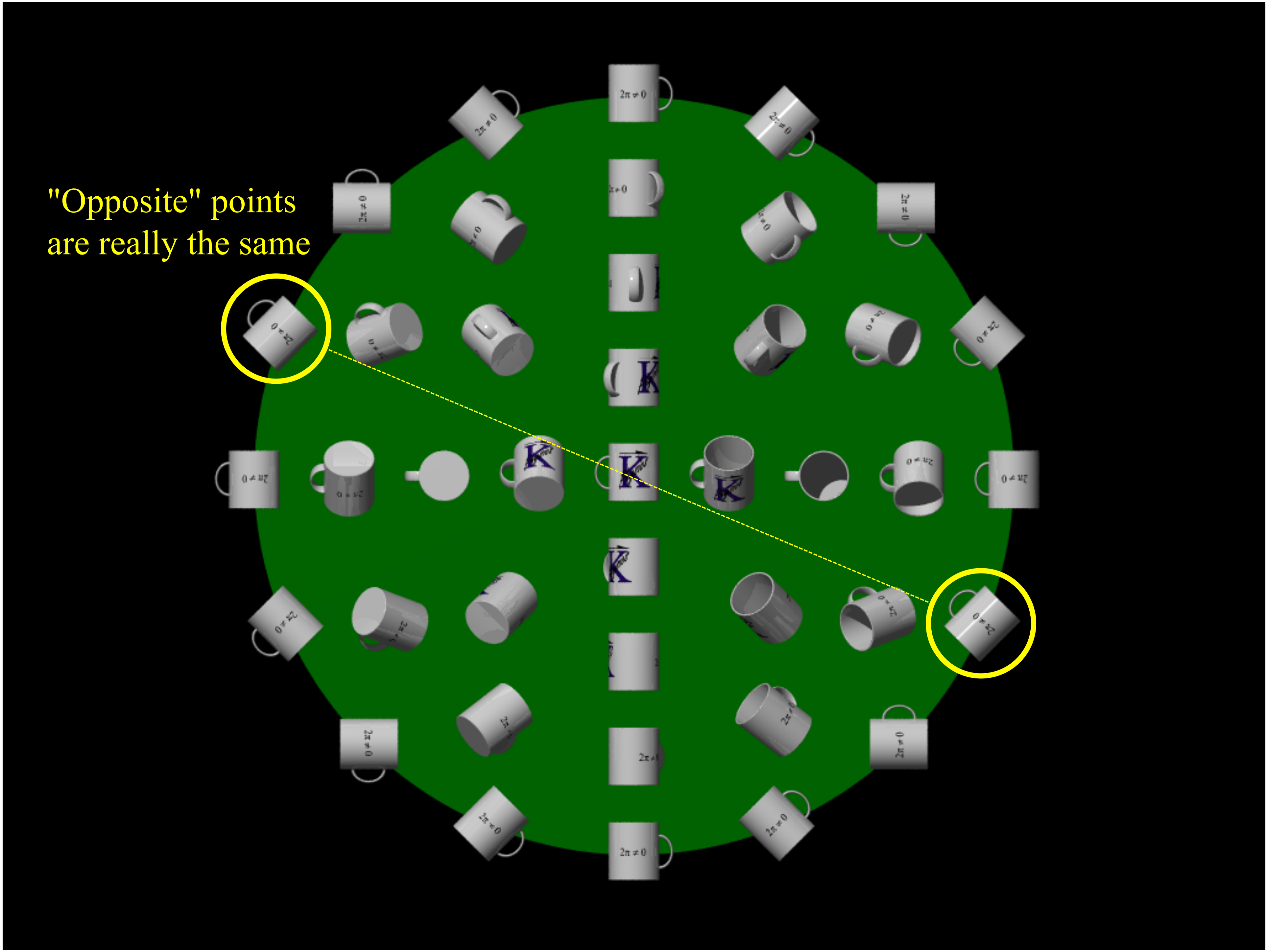


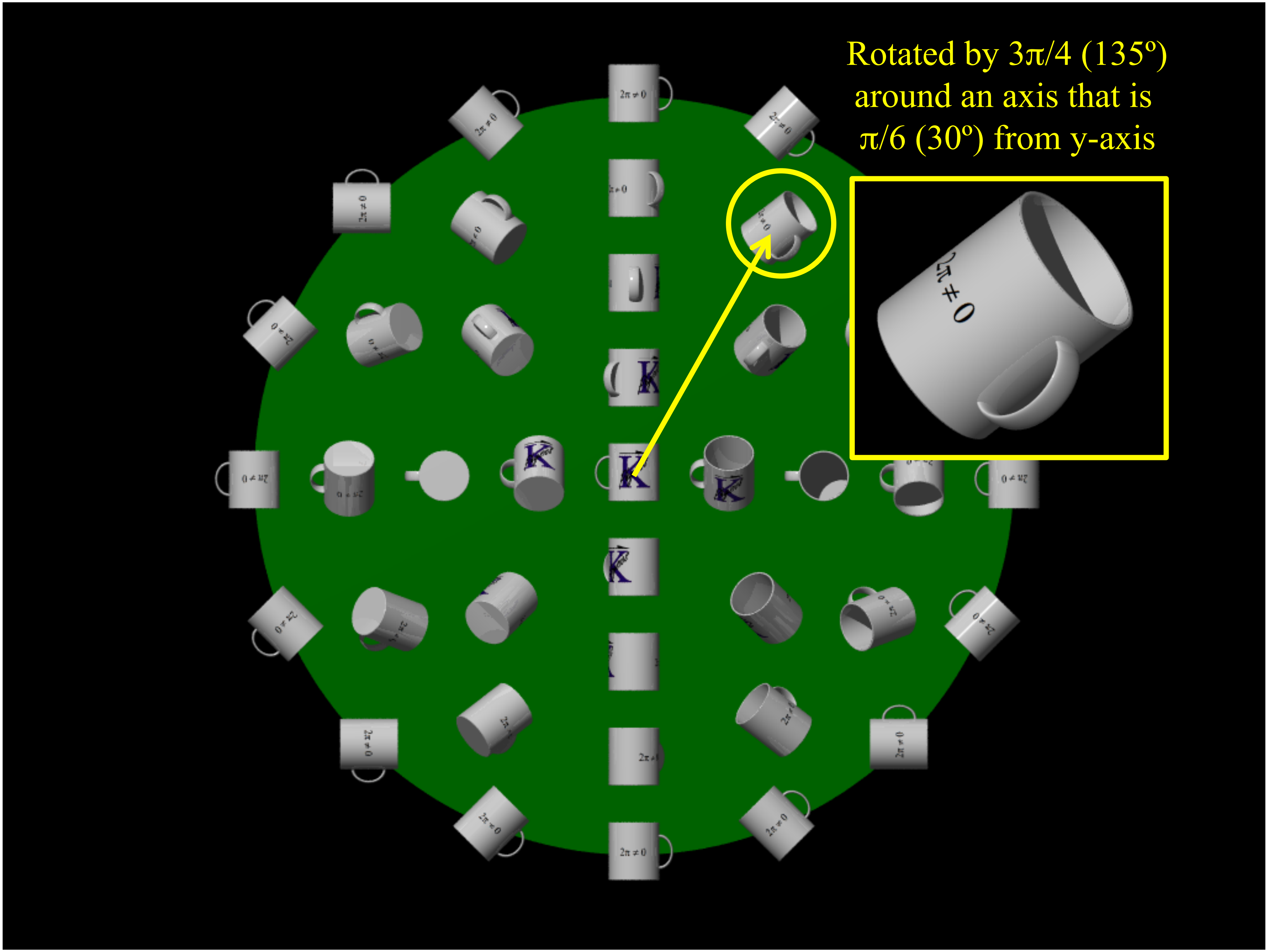
Rotated around the y-axis.

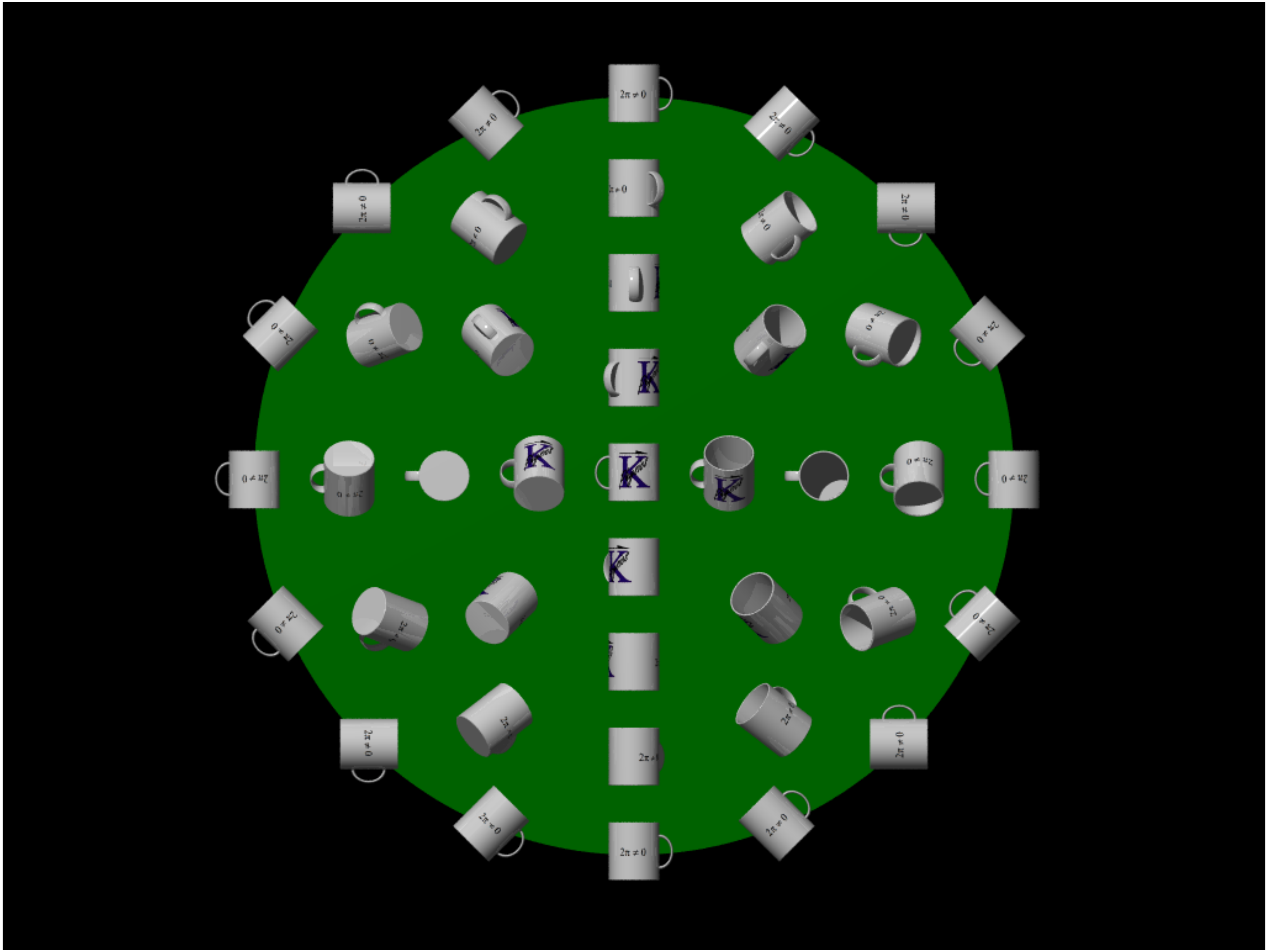


Rotated by an angle of π radians

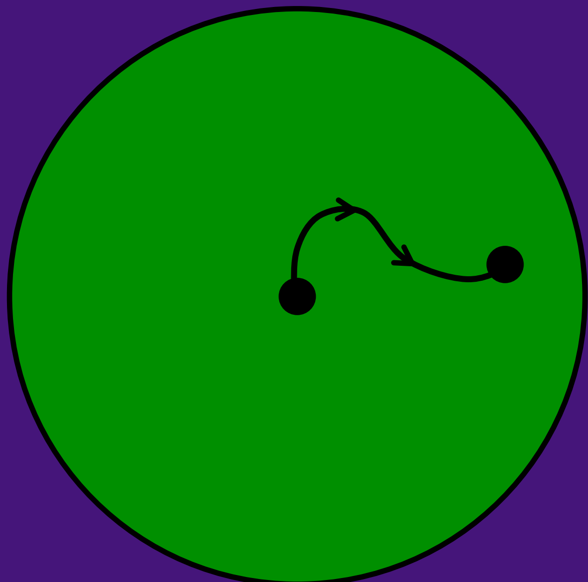
(Note that all of these images show the "back side" of the mug.)


"Opposite" points
are really the same


Rotated by 3π/4 (135º)
around an axis that is
π/6 (30º) from y-axis
2π ≠ 0


2π ≠ 0

Journeys through O-space



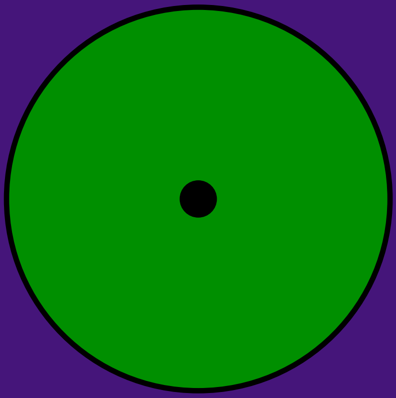
- When we rotate an object, it follows a continuous path through O-space.
- A "complete rotation" is a closed path -- one that begins and ends at the center point.
- Our mystery: Not all closed paths are the same!

Going "once around" is not like staying in one place, but going "twice around" is.

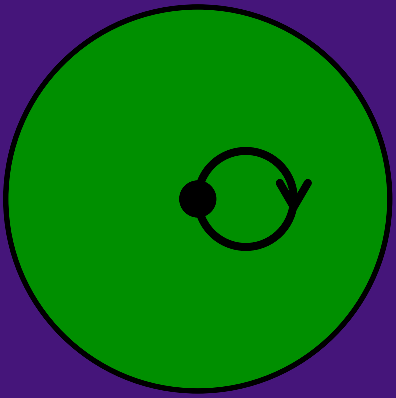
Some closed paths

Just sit there

A slight wobble



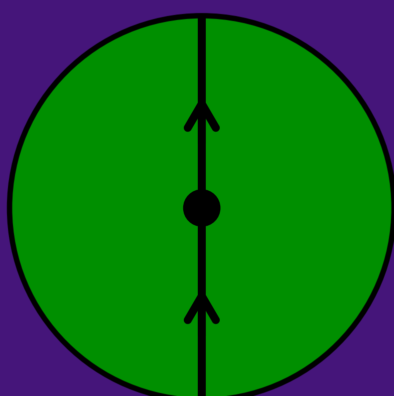
Sometimes called "path 0"

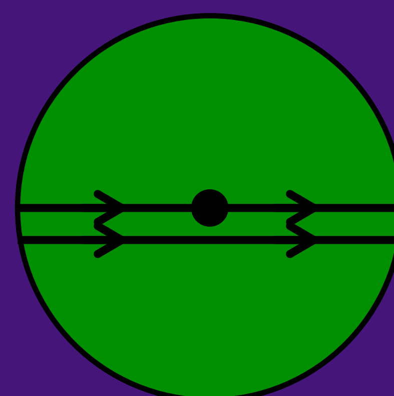


Some closed paths

Once around y-axis

Twice around x-axis



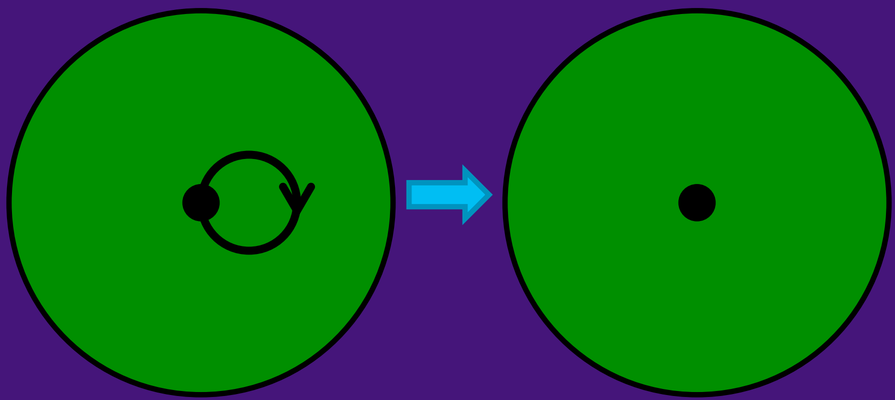


Paths that are almost alike

Two paths are similar ("homotopic") if

- One path is a small alteration of the other.
- One path can be turned into the other by a series of small alterations.

Example: A wobble path is homotopic to 0

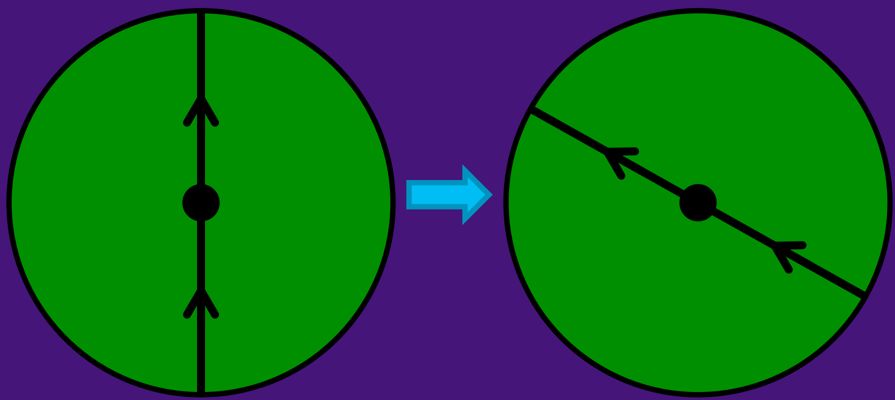


Paths that are almost alike

Two paths are similar (homotopic) if

- One path is a small alteration of the other.
- One path can be turned into the other by a series of small alterations.

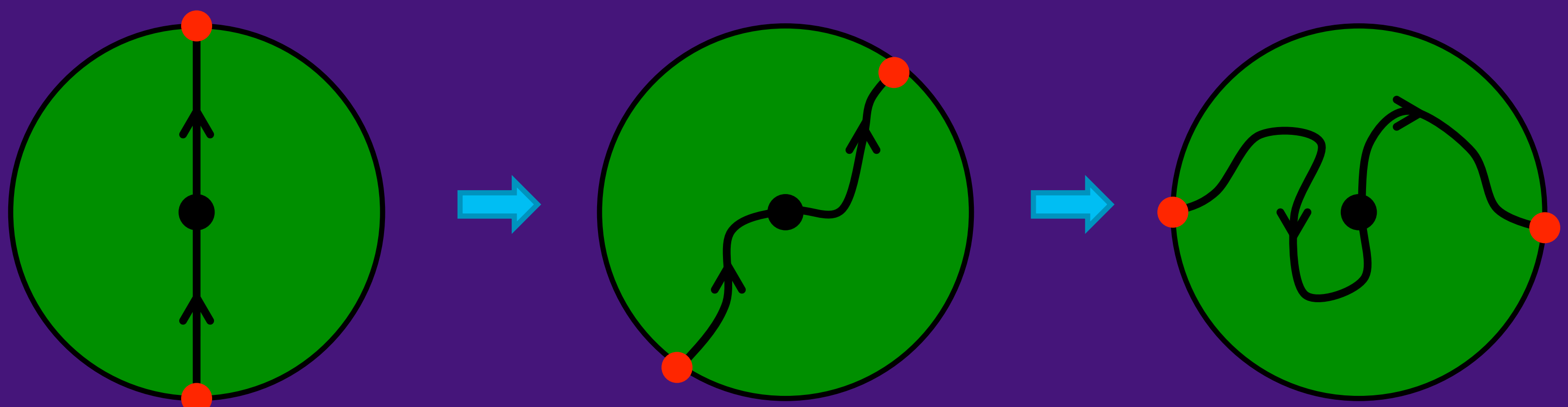
Example: Once around y-axis is homotopic to once around any axis



Once around is *not* like 0

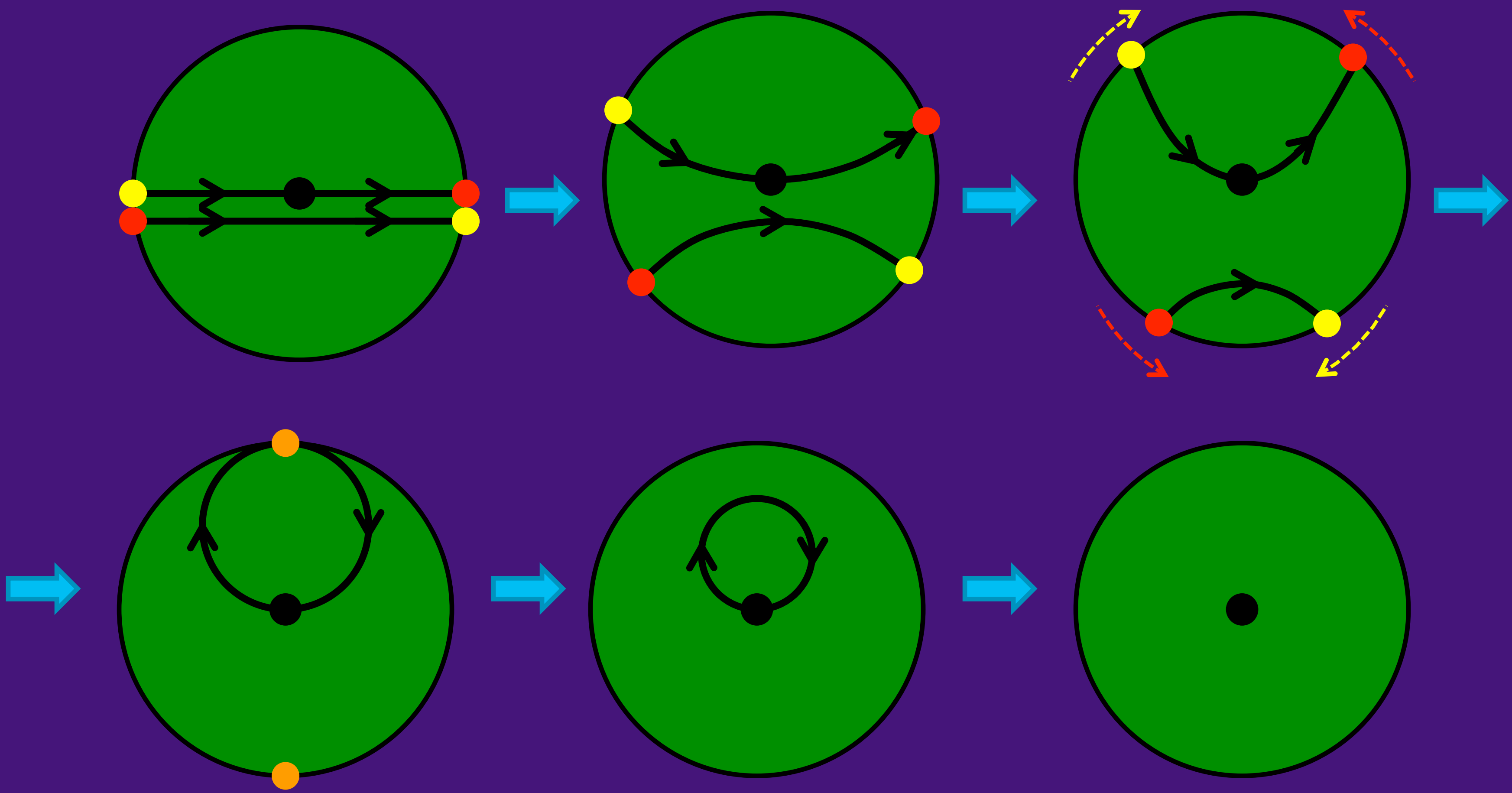
Fact: Once around y-axis (or any axis) is not similar to path 0.

Why: No matter how we tweak the path, it still touches the outer edge in at least two opposite points.



Twice around *is* like 0

Fact: Twice around any axis is homotopic to path 0.



The meaning of the minus

Recall quantum spin ...

Rotating a spin-1/2 particle by 2π yields a weird minus sign.

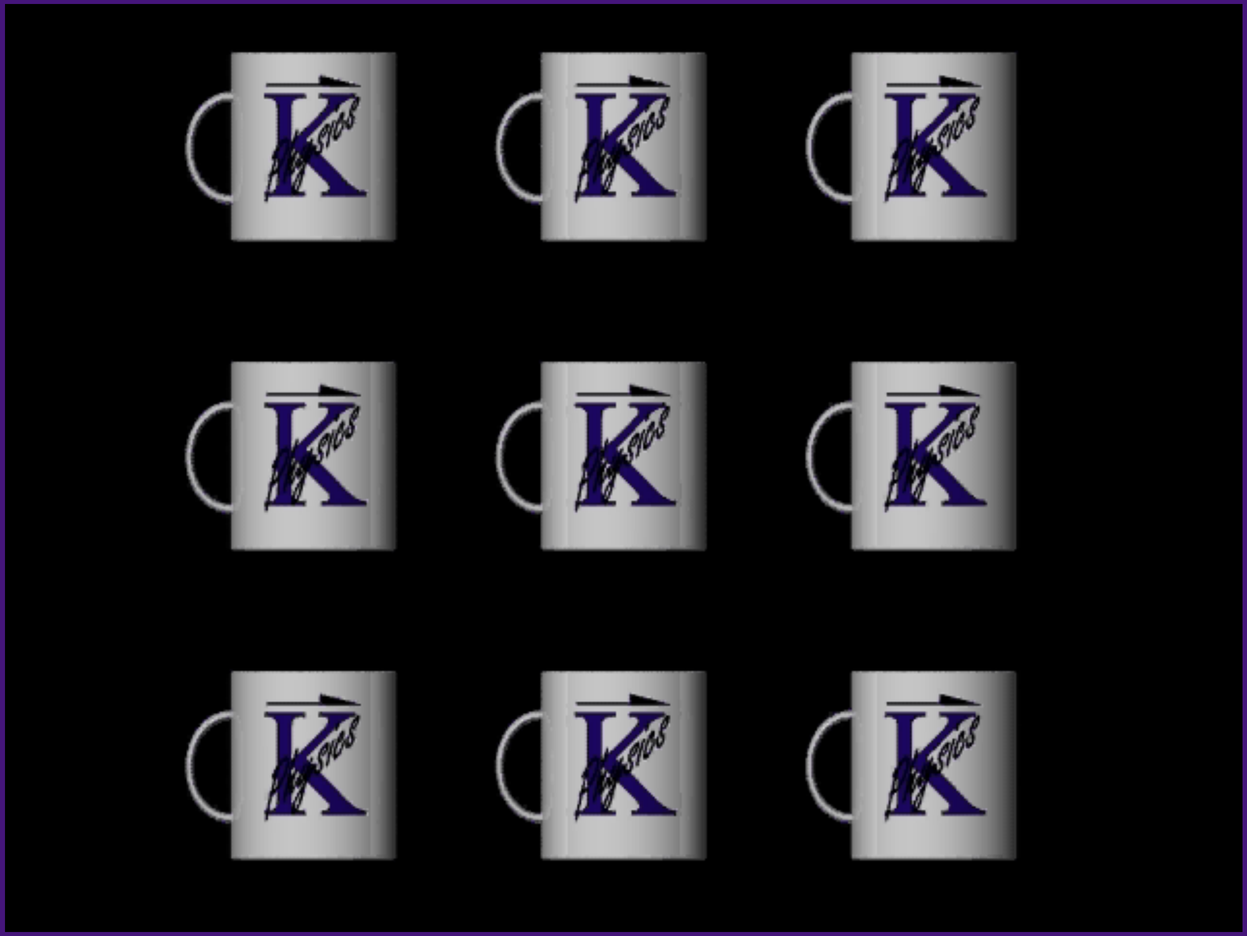
$$\mathrm{Rot}(2\pi)|\phi\rangle = -|\phi\rangle$$
$$\mathrm{Rot}(4\pi)|\phi\rangle = +|\phi\rangle$$

Minus sign means that our "once around" (2π) closed path through O-space is not homotopic to path 0 (no rotation).

"Twice around" (4π) path can be changed to path 0 in a continuous way, so it is homotopic to 0 (no minus sign).

OK, but what does this look like?

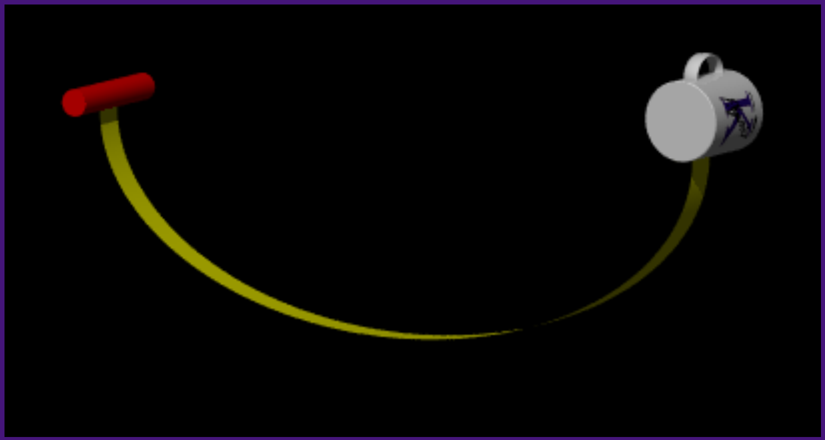
Nine mug dance



Rotation is relational

- Rotating an object changes its "orientation relationship" with the rest of the Universe.
- Keeping track of the relationship: A connecting ribbon!

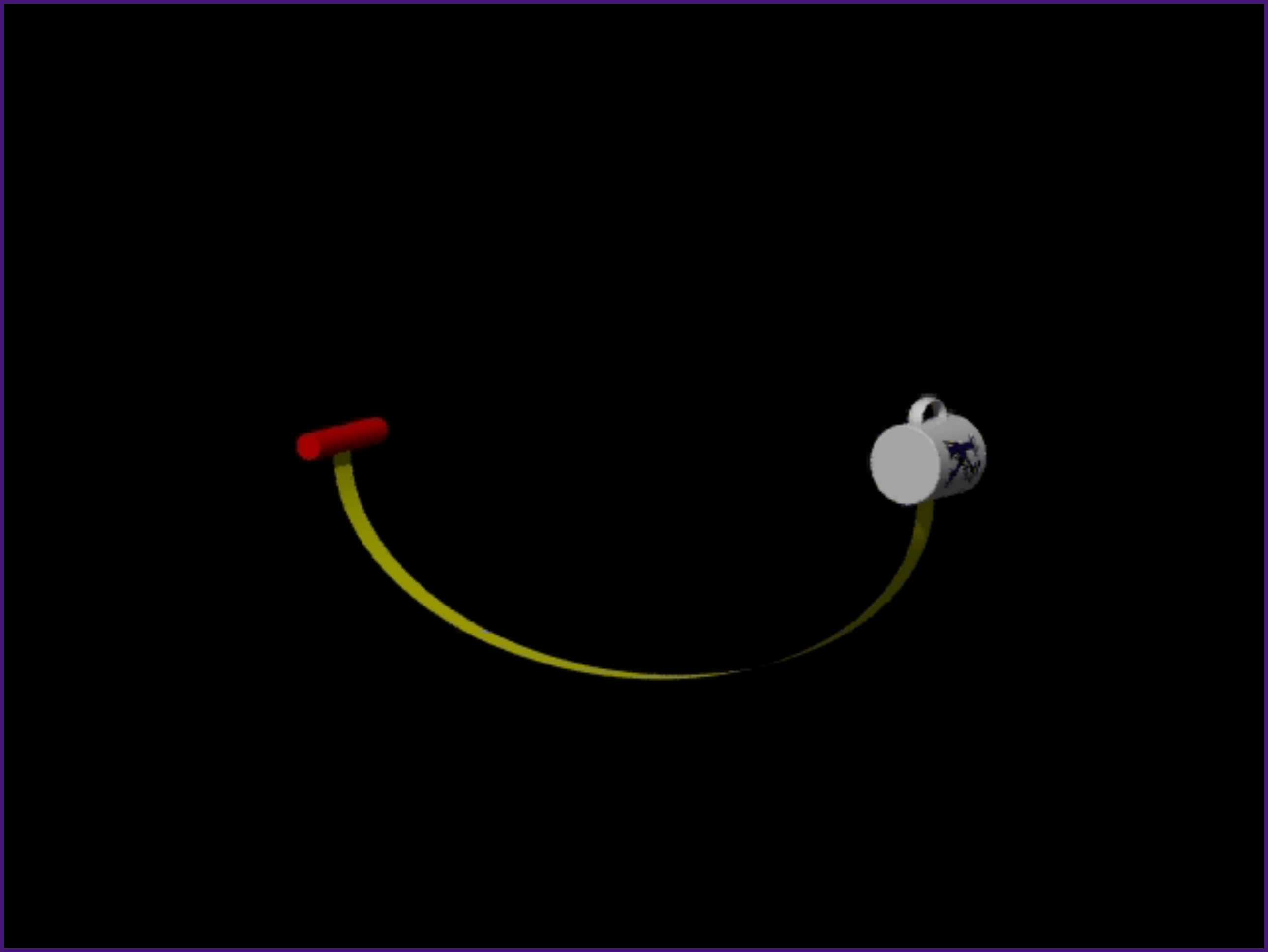
Fixed red rod (the rest of the Universe)



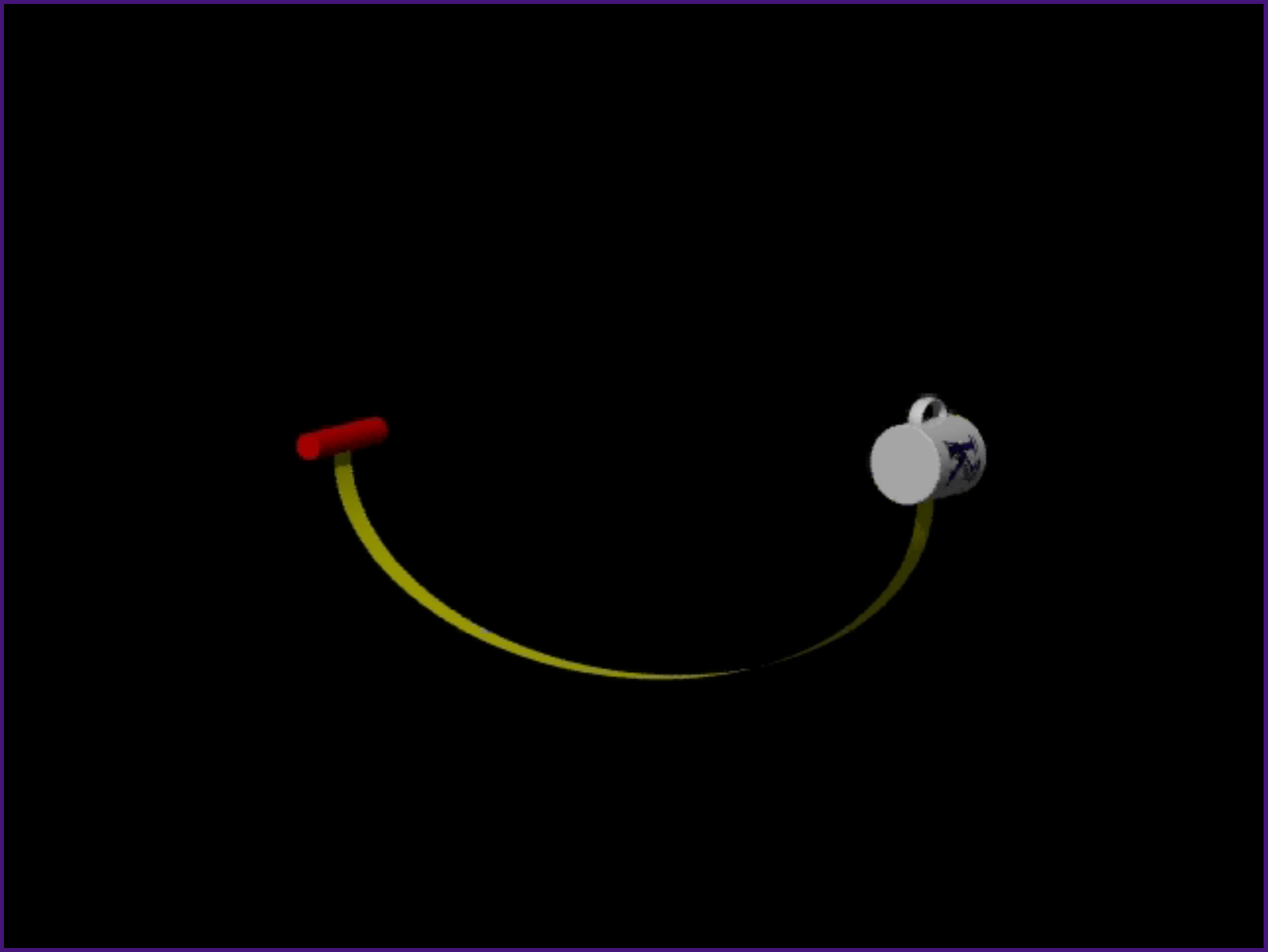
Rotating object with ribbon attached

Flexible ribbon represents the orientation relationship

Rot(2π) twists the ribbon



Rot(4π) untwists the ribbon



Doing it yourself

- You can do this demonstration with a ribbon or a belt.

  One turn always yields a twist.

  Two turns yields no net twist.

- Awkward staging -- ribbon needs to pass around one end.
- Can also be done with a coffee mug (preferably empty).

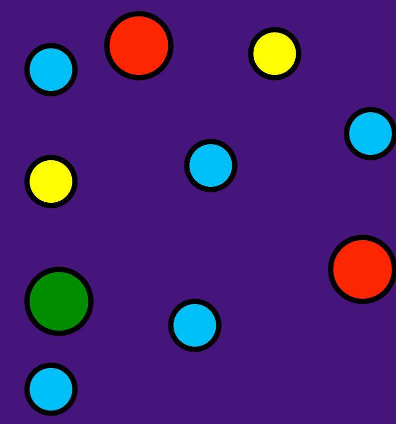
What the twist tells us:

2π is not zero (but 4π is)!

Another weird quantum minus sign

Identical particles

- Quantum particles can be exactly identical.
- All electrons are exactly the same (no physical "serial numbers")
- If we exchange any two electrons, the new situation looks just like the old one.



Curious quantum fact: If we exchange two electrons, we end up with a minus sign!

$$X(1,2)\left|\Psi\right\rangle = -\left|\Psi\right\rangle$$

Is it important?

- This is possibly the most important minus sign in all of physics!
- Pauli exclusion principle: No two electrons can be in the same quantum state (e.g., same location with the same spin)

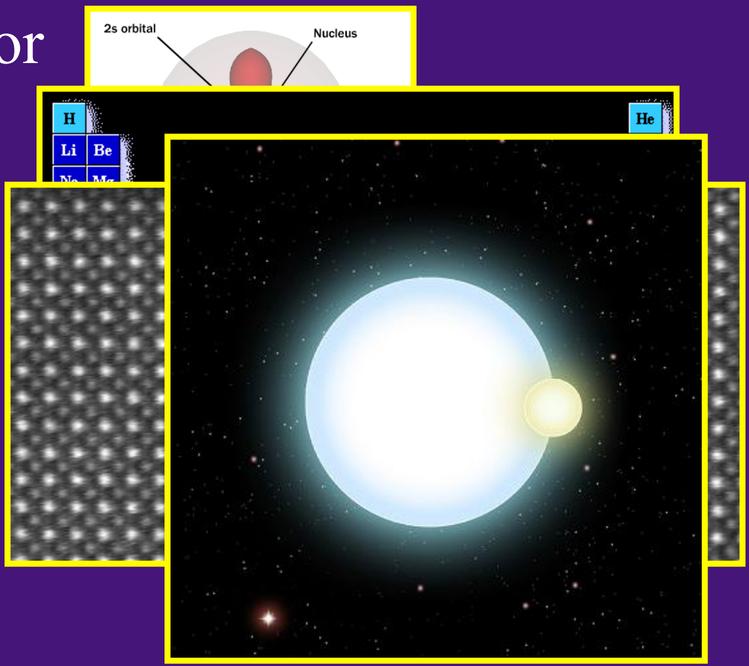
If 1 and 2 were in the same state, then $|\Psi(1,2)\rangle = |\Psi(2,1)\rangle$

But $|\Psi(2,1)\rangle = X|\Psi(1,2)\rangle = -|\Psi(1,2)\rangle$

Thus $|\Psi(1,2)\rangle = -|\Psi(1,2)\rangle = 0$ *Impossible!*

Is it important?

- This is possibly the most important minus sign in all of physics!
- Pauli exclusion principle: No two electrons can be in the same quantum state (e.g., same location with the same spin)
- This fact is ultimately responsible for
    - Electron structure of atoms
    - All chemical properties
    - Why matter "takes up space"
    - Structure of collapsed stars
    - Etc.



Two types of particles

Fermions

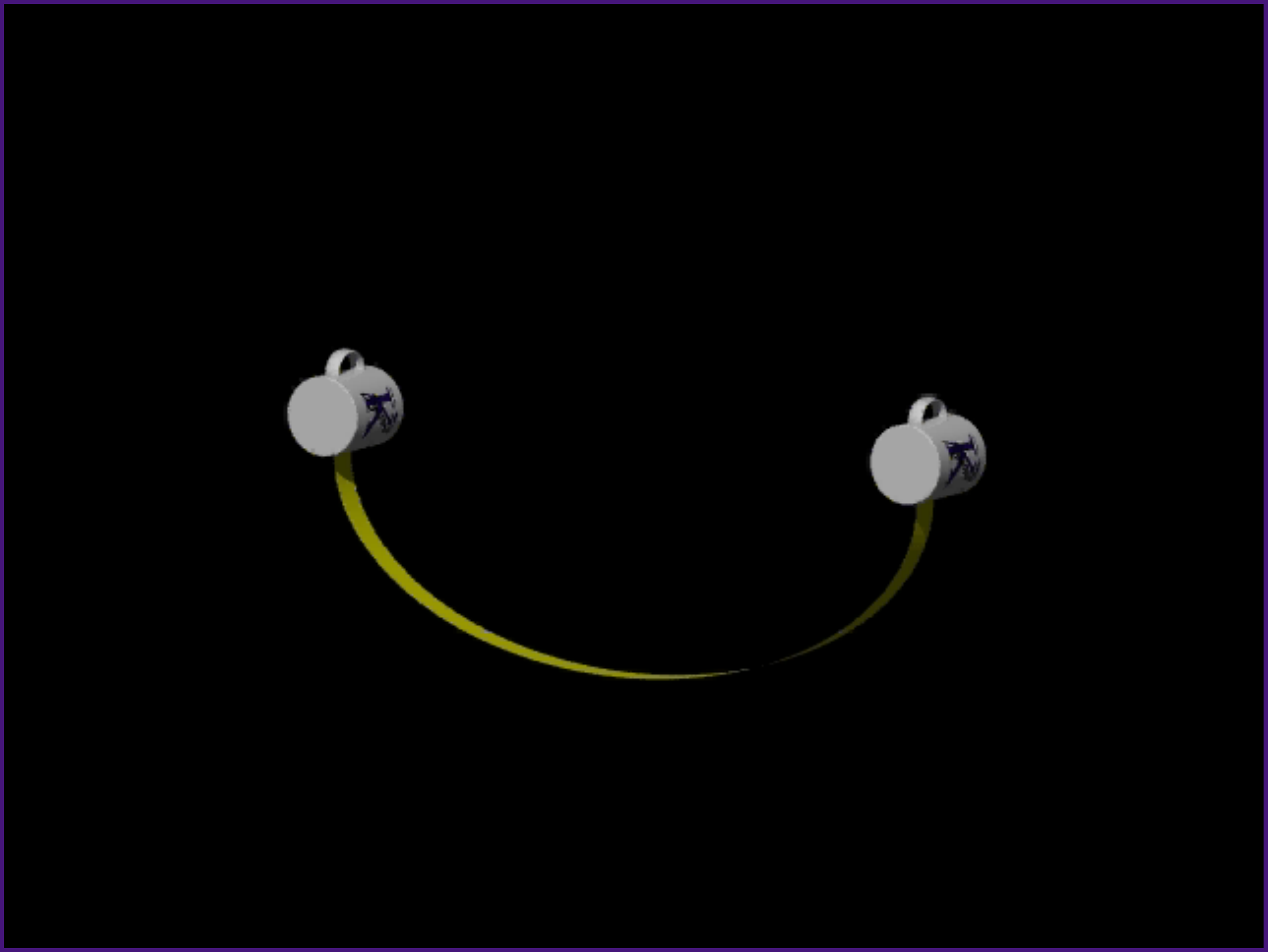
- Electrons, protons, neutrons, etc.
- $X(1,2)|\Psi\rangle = -|\Psi\rangle$
- Obey Pauli exclusion principle
- Spin 1/2, 3/2, ...
- Rot(2π) = −1

Bosons

- Photons, ^{4}He atoms, Cooper pairs
- $X(1,2)|\Psi\rangle = +|\Psi\rangle$
- Do not obey Pauli exclusion principle
- Spin 0, 1, 2, ...
- Rot(2π) = +1

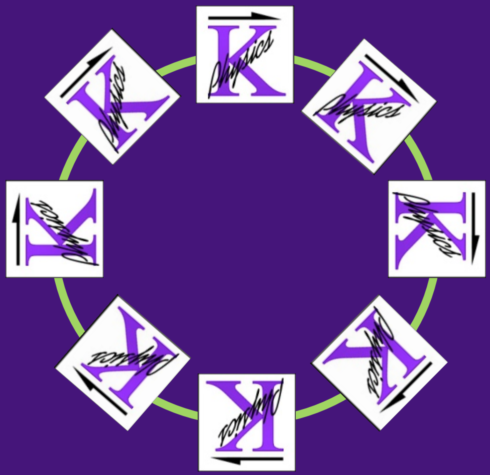
Is there a link between rotation and particle exchange?

Mug exchange



How about a 2-D world?

- In a 2-D world, O-space only has one dimension!



- Rot(2π) is not zero, but neither are Rot(4π), Rot(6π), etc.
- Quantum physics: Other possibilities besides fermions and bosons -- generically known as anyons.

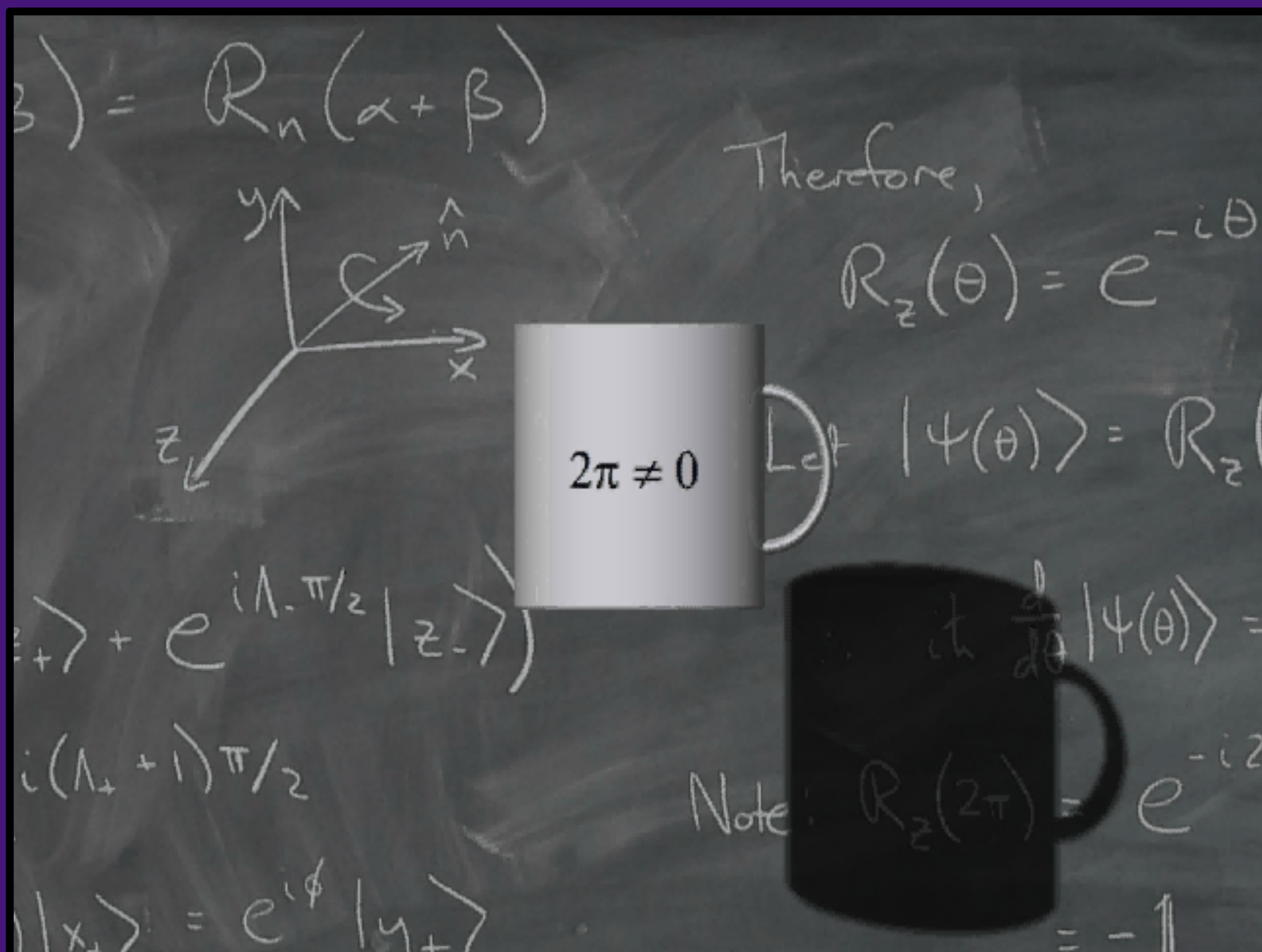
Places we've been

- The quantum physics of spin-1/2 particles forces us to introduce a strange minus sign in 2π rotation. This has actual experimental consequences.
- A rotation is a closed path in O-space. Not all closed paths are homotopic to path 0 (no rotation).
- Ribbon model: A rotation of 2π introduces a twist, but a rotation of 4π does not.
- Minus sign in rotation is the same as the minus sign in fermion particle exchange -- the most important minus sign in the universe!

Things we *didn't* say
(*and don't you feel lucky*)

- Rotation operators generated by angular momentum
- Group homomorphism: SU(2) → SO(3) is 2-to-1
- O-space is the group manifold of SO(3)
- The fundamental group of the SO(3) manifold is Z_2
- Symmetric and antisymmetric quantum states
- Fiertz and Pauli (1940): Spin-statistics theorem in quantum field theory

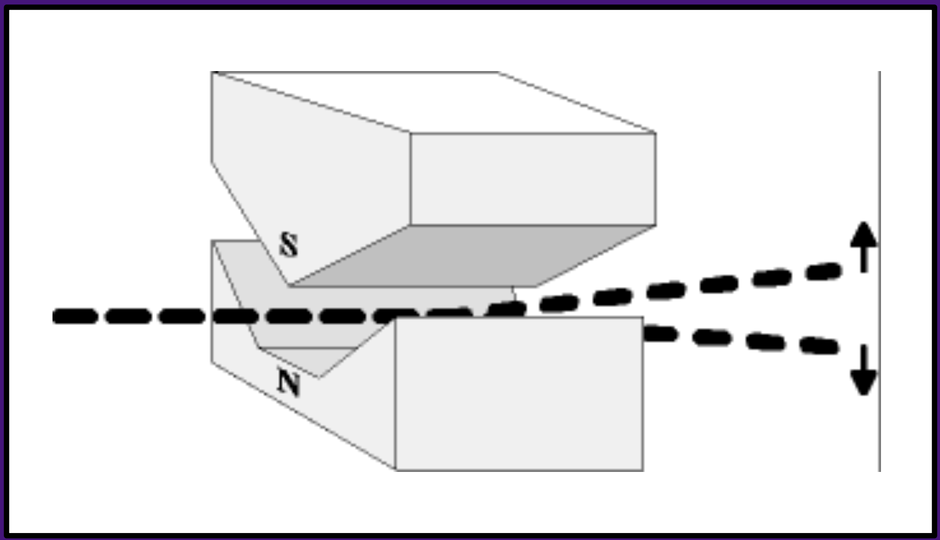
The End



Things to read, watch, play with

- *Richard Feynman: "The reason for antiparticles" (1986 Dirac Memorial Lecture).*
- *Bob Palais, Richard Palais, and Stephen Rodi "A Disorienting Look at Euler's Theorem on the Axis of a Rotation" American Mathematical Monthly (2009).*
- *BWS: "Quantum Mechanics: The Physics of the Microscopic World" (The Great Courses, 2009) -- especially lectures 10-14.*
- *POVray: Persistence of Vision ray-tracing program, augmented by Maple, C++, video editing software, etc.*

Spin up, spin down



- Spin-1/2 particles
- We measure S_z (one component of spin)
- Possible results:

  $+\hbar/2$ (up) or $-\hbar/2$ (down)

Quantum states:

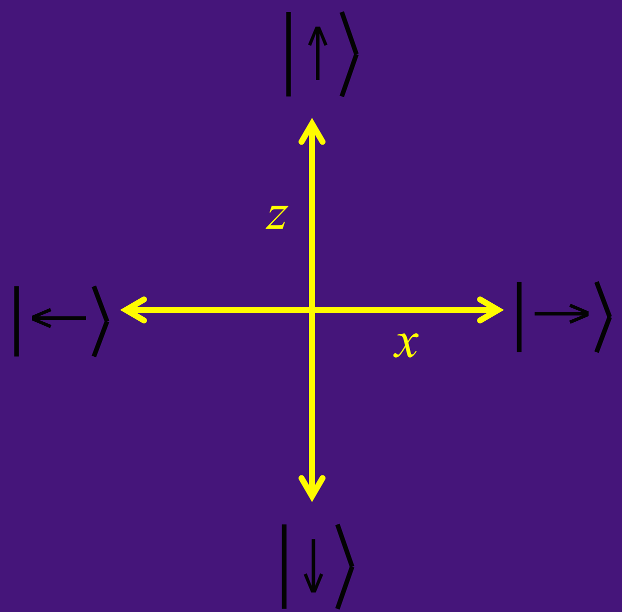
$$|\uparrow\rangle,\ |\downarrow\rangle,\ |\phi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$$

superposition state

$$P(\uparrow) = |a|^2$$

$$P(\downarrow) = |b|^2$$

Spin right, spin left



Any spin state can be built out of $|\uparrow\rangle$ and $|\downarrow\rangle$

$$|\rightarrow\rangle = s|\uparrow\rangle + s|\downarrow\rangle$$

$$|\leftarrow\rangle = s|\uparrow\rangle - s|\downarrow\rangle$$

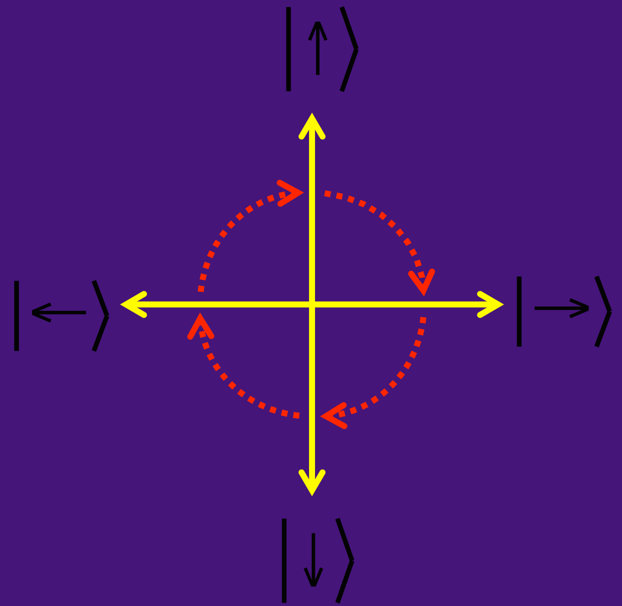
$$s = \frac{1}{\sqrt{2}}$$

To "rotate" a spin state, apply rotation operator *R*:

$$\begin{aligned} |\phi'\rangle &= R|\phi\rangle \\ &= R\left(a|\uparrow\rangle + b|\downarrow\rangle\right) \\ &= a\,R|\uparrow\rangle + b\,R|\downarrow\rangle \end{aligned}$$

(*R* operator depends on axis and angle of rotation.)

Rotating by π/2



Rotate about the y-axis. We'd like:

$$R|\uparrow\rangle = |\rightarrow\rangle \quad R|\rightarrow\rangle = |\downarrow\rangle$$
$$R|\downarrow\rangle = |\leftarrow\rangle \quad R|\leftarrow\rangle = |\uparrow\rangle$$

But this is not possible!

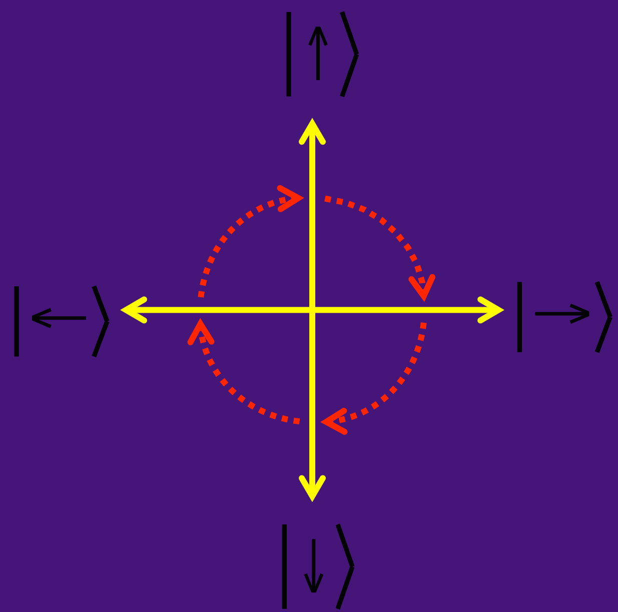
Suppose $R|\uparrow\rangle = |\rightarrow\rangle = s|\uparrow\rangle + s|\downarrow\rangle$

$R|\downarrow\rangle = |\leftarrow\rangle = s|\uparrow\rangle - s|\downarrow\rangle$

Then
$$R|\rightarrow\rangle = sR|\uparrow\rangle + sR|\downarrow\rangle$$
$$= (s^2 + s^2)|\uparrow\rangle + (s^2 - s^2)|\downarrow\rangle$$
$$= |\uparrow\rangle$$

Uh-oh.

Rotating by π/2



How do we fix this? Try:

$$R|\uparrow\rangle = |\rightarrow\rangle = s|\uparrow\rangle + s|\downarrow\rangle$$

$$R|\downarrow\rangle = \alpha|\leftarrow\rangle = \alpha\big(s|\uparrow\rangle - s|\downarrow\rangle\big)$$

Then $R|\rightarrow\rangle = sR|\uparrow\rangle + sR|\downarrow\rangle$

$$= (s^2 + \alpha s^2)|\uparrow\rangle + (s^2 - \alpha s^2)|\downarrow\rangle$$

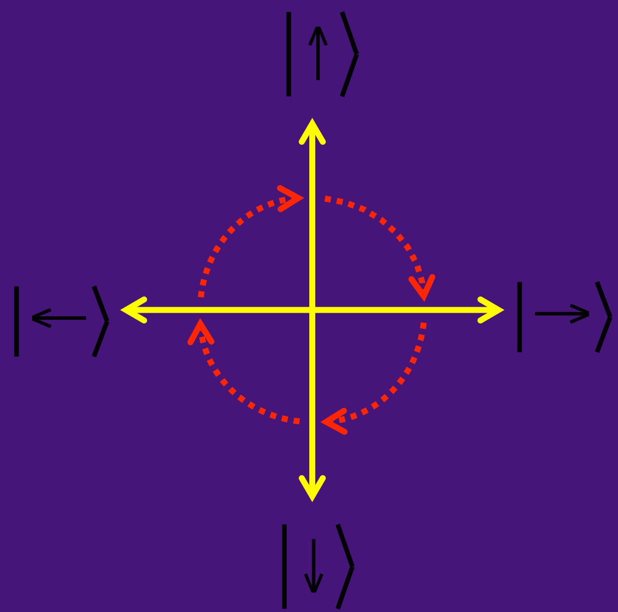
$$= |\downarrow\rangle \quad (\text{provided } \alpha = -1)$$

$$R|\uparrow\rangle = |\rightarrow\rangle \qquad R|\rightarrow\rangle = |\downarrow\rangle$$

$$R|\downarrow\rangle = -|\leftarrow\rangle \qquad R|\leftarrow\rangle = |\uparrow\rangle$$

The actual rotation rule

Rotating by 2π



$$R|\uparrow\rangle = |\rightarrow\rangle \qquad R|\rightarrow\rangle = |\downarrow\rangle$$
$$R|\downarrow\rangle = -|\leftarrow\rangle \qquad R|\leftarrow\rangle = |\uparrow\rangle$$

The actual rotation rule

A peculiar minus sign:

$$\mathrm{Rot}(2\pi) = R \cdot R \cdot R \cdot R = R^4$$
$$|\phi\rangle \rightarrow R^4|\phi\rangle = -|\phi\rangle$$

Rot(2π) ≠Rot(0), but
Rot(4π) = Rot(0)

Spin-1/2 particles see
a "4π" world!

Rot(2π) ≠ Rot(0) but Rot(4π) = Rot(0)

First thought

This is totally weird. How can this be right??

Second thought

Maybe this is not so bad. Minus sign is unobservable!

$$|\phi\rangle = a|\uparrow\rangle + b|\downarrow\rangle \Rightarrow P(\uparrow) = |a|^2 = |-a|^2$$

$$P(\downarrow) = |b|^2 = |-b|^2$$

Probabilities don't change if $|\phi\rangle \to -|\phi\rangle$

Third thought

On the other hand ....