$2\pi$ is not zero (but $4\pi$ is)
How mathematicians think about angle: the unit circle

Angle measure in "radians": length of arc on unit circle

Circumference = \(2\pi \times \text{radius}\)

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Radians</th>
<th>What it means</th>
</tr>
</thead>
<tbody>
<tr>
<td>90°</td>
<td>(\pi/2)</td>
<td>A quarter-turn</td>
</tr>
<tr>
<td>180°</td>
<td>(\pi)</td>
<td>Halfway around</td>
</tr>
<tr>
<td>360°</td>
<td>(2\pi)</td>
<td>One complete turn</td>
</tr>
<tr>
<td>720°</td>
<td>(4\pi)</td>
<td>Two complete turns</td>
</tr>
</tbody>
</table>
Of course $2\pi = 0!$

If we rotate a geometrical shape by $2\pi$ radians (360º), there is no net change to the shape.

Also true in 3-D space:

- any shape
- any axis of rotation
- $\text{Rot}(2\pi) = \text{Rot}(0) = 1$
A quantum puzzle
Quantum states

• Physical situation described by a mathematical object: the quantum state $|\Psi\rangle$

• Some quantum states describe familiar situations: $|\text{cat alive}\rangle$, $|\text{cat dead}\rangle$

• Objects can also be in a "superposition" state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} |\text{cat alive}\rangle + \frac{1}{\sqrt{2}} |\text{cat dead}\rangle$$

Observe the cat:

$$P(\text{alive}) = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$
$$P(\text{dead}) = \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2}$$
Quantum spin

- Particles have an "internal" angular momentum called spin.
- Total spin can be 0, 1/2, 1, 3/2, etc. (in units of ħ).
- Electrons, protons and neutrons have spin 1/2.
- Spin is related to magnetic properties -- we can affect and measure spin using magnetic fields.
A curious fact about rotation

• Spin-1/2 particle
• Start with any spin state $|\phi\rangle$
• Rotate spin by $2\pi$ (360°): $|\phi\rangle \rightarrow \text{Rot}(2\pi)|\phi\rangle = -|\phi\rangle$

• In effect, Rot($2\pi$) = -1, not +1
• To return to the initial quantum state, we must rotate the spin by $4\pi$ (720°). Rot($4\pi$) = +1!

Does this fact have any observable consequences?
(Some books say "no" -- but they're wrong.)
Neutron interferometry

Quantum physics: Neutrons travel through space as waves.
We can arrange for these waves to interfere with each other.

Neutron beam enters here

Neutrons can follow two possible paths

Beams recombine

Neutrons are detected at one detector or the other.

Si single-crystal neutron interferometers at NIST
Neutron interferometry

Constructive interference: Waves add up to a more intense wave

Destructive interference: Waves cancel out to zero
Neutron interferometry

Use magnetic field to rotate neutron spins in lower beam by $2\pi$

Note: If we rotate by $4\pi$, we get back the original pattern.

Relative minus sign changes constructive to destructive interference and *vice versa* -- can be (and is) observed!

Rot($2\pi$) is not the same as Rot(0), but Rot($4\pi$) is!
A visit to O-space
Orientation

• When we rotate an object, we change its orientation in space.

• Think of this as a kind of "motion" in "orientation space" (O-space).

• Each possible orientation of the object is a "point" in O-space.

• What does O-space look like for 3-D objects?
Euler's rotation theorem

Any point in O-space can be reached by
• choosing an axis in space, and
• rotating about that axis by some angle.

Rotation direction is given by "right-hand rule".
Rotation angle is between 0 and π.
A map of O-space

O-space is a sphere of radius \( \pi \).

Antipodal points on the sphere are really the same point in O-space.

Direction indicates axis.
Distance from center indicates angle (0 to \( \pi \)).

It's hard to think in 3-D -- let's consider the "xy-slice" across O-space.
Unrotated mug at the center
Rotated around the x-axis.
Rotated around the y-axis.
Rotated by an angle of $\pi$ radians

(Note that all of these images show the "back side" of the mug.)
"Opposite" points are really the same
Rotated by $3\pi/4$ (135°) around an axis that is $\pi/6$ (30°) from y-axis.
Journeys through O-space

• When we rotate an object, it follows a **continuous path** through O-space.

• A "complete rotation" is a **closed path** -- one that begins and ends at the center point.

• Our mystery: **Not all closed paths are the same**!

*Going "once around" is not like staying in one place, but going "twice around" is.*
Some closed paths

Just sit there

A slight wobble

Sometimes called "path 0"
Some closed paths

Once around y-axis

Twice around x-axis
Paths that are almost alike

Two paths are similar ("homotopic") if

- One path is a **small alteration** of the other.
- One path can be turned into the other by a **series of small alterations**.

Example: A wobble path is homotopic to 0
Paths that are almost alike

Two paths are similar (homotopic) if

• One path is a small alteration of the other.

• One path can be turned into the other by a series of small alterations.

Example: Once around y-axis is homotopic to once around any axis
Once around is *not* like 0

Fact: Once around y-axis (or any axis) is not similar to path 0.

Why: No matter how we tweak the path, it still touches the outer edge in at least two opposite points.
Twice around is like 0

Fact: Twice around any axis is homotopic to path 0.
The meaning of the minus

Recall quantum spin ...

Rotating a spin-1/2 particle by $2\pi$ yields a weird minus sign.

$$\text{Rot}(2\pi)|\phi\rangle = -|\phi\rangle$$
$$\text{Rot}(4\pi)|\phi\rangle = +|\phi\rangle$$

Minus sign means that our "once around" ($2\pi$) closed path through O-space is not homotopic to path 0 (no rotation).

"Twice around" ($4\pi$) path can be changed to path 0 in a continuous way, so it is homotopic to 0 (no minus sign).

_OK, but what does this look like?_
Nine mug dance
Rotation is relational

• Rotating an object changes its "orientation relationship" with the rest of the Universe.

• Keeping track of the relationship: A connecting ribbon!

Fixed red rod (the rest of the Universe)  Rotating object with ribbon attached

Flexible ribbon represents the orientation relationship
Rot($2\pi$) twists the ribbon
Rot($4\pi$) untwists the ribbon
Doing it yourself

- You can do this demonstration with a ribbon or a belt.
  - One turn always yields a twist.
  - Two turns yields no net twist.
- Awkward staging -- ribbon needs to pass around one end.
- Can also be done with a coffee mug (preferably empty).

What the twist tells us:

\[ 2\pi \text{ is not zero (but } 4\pi \text{ is)}! \]
Another weird quantum minus sign
Identical particles

• Quantum particles can be exactly identical.

• All electrons are exactly the same (no physical "serial numbers")

• If we exchange any two electrons, the new situation looks just like the old one.

Curious quantum fact: If we exchange two electrons, we end up with a minus sign!

$$X(1,2) |\Psi\rangle = -|\Psi\rangle$$
Is it important?

• This is possibly the most important minus sign in all of physics!

• Pauli exclusion principle: No two electrons can be in the same quantum state (e.g., same location with the same spin)

If 1 and 2 were in the same state, then $|\Psi(1,2)\rangle = |\Psi(2,1)\rangle$

But $|\Psi(2,1)\rangle = X|\Psi(1,2)\rangle = -|\Psi(1,2)\rangle$

Thus $|\Psi(1,2)\rangle = -|\Psi(1,2)\rangle = 0$  Impossible!
Is it important?

• This is possibly the most important minus sign in all of physics!

• **Pauli exclusion principle**: No two electrons can be in the same quantum state (e.g., same location with the same spin)

• This fact is ultimately responsible for
  
  • Electron structure of atoms  
  • All chemical properties  
  • Why matter "takes up space"  
  • Structure of collapsed stars  
  • Etc.
Two types of particles

**Fermions**
- Electrons, protons, neutrons, etc.
- $X(1,2)|\Psi\rangle = -|\Psi\rangle$
- Obey Pauli exclusion principle
- Spin 1/2, 3/2, ....
- $\text{Rot}(2\pi) = -1$

**Bosons**
- Photons, $^4\text{He}$ atoms, Cooper pairs
- $X(1,2)|\Psi\rangle = +|\Psi\rangle$
- Do not obey Pauli exclusion principle
- Spin 0, 1, 2, ....
- $\text{Rot}(2\pi) = +1$

Is there a link between rotation and particle exchange?
Mug exchange
How about a 2-D world?

• In a 2-D world, O-space only has one dimension!

• Rot(2\pi) is not zero, but neither are Rot(4\pi), Rot(6\pi), etc.

• Quantum physics: Other possibilities besides fermions and bosons -- generically known as anyons.
Places we've been

• The quantum physics of spin-1/2 particles forces us to introduce a **strange minus sign** in $2\pi$ rotation. This has actual **experimental consequences**.

• A rotation is a closed path in **O-space**. Not all closed paths are **homotopic** to path 0 (no rotation).

• Ribbon model: A rotation of $2\pi$ introduces a **twist**, but a rotation of $4\pi$ does not.

• Minus sign in rotation is the same as the minus sign in **fermion particle exchange** -- the most important minus sign in the universe!
Things we didn't say (and don't you feel lucky)

• Rotation operators generated by angular momentum
• Group homomorphism: $\text{SU}(2) \rightarrow \text{SO}(3)$ is 2-to-1
• O-space is the group manifold of $\text{SO}(3)$
• The fundamental group of the $\text{SO}(3)$ manifold is $\mathbb{Z}_2$
• Symmetric and antisymmetric quantum states
• Fiertz and Pauli (1940): Spin-statistics theorem in quantum field theory
The End

\[ |\psi\rangle = R_n(x + \beta) |\psi\rangle \]

Therefore,

\[ R_z(\theta) = e^{-i\theta} \]

Let \[ |\psi(\theta)\rangle = R_z |\psi\rangle \]

Note: \[ R_z(2\pi) = e^{i(\lambda_1 + \lambda_2 + \pi/2)} \]

\[ = e^{-i(\lambda_1 + \lambda_2 + \pi/2)} \]

\[ = -1 \]
Things to read, watch, play with

• Richard Feynman: "The reason for antiparticles" (1986 Dirac Memorial Lecture).


• POVray: Persistence of Vision ray-tracing program, augmented by Maple, C++, video editing software, etc.
Spin up, spin down

- Spin-1/2 particles
- We measure $S_z$ (one component of spin)
- Possible results:
  $+\hbar/2$ (up) or $-\hbar/2$ (down)

Quantum states:

$|\uparrow\rangle$, $|\downarrow\rangle$, $|\phi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$

$p(\uparrow) = |a|^2$

$p(\downarrow) = |b|^2$

superposition state
Spin right, spin left

Any spin state can be built out of $|\uparrow\rangle$ and $|\downarrow\rangle$

$$|\rightarrow\rangle = s|\uparrow\rangle + s|\downarrow\rangle$$
$$|\leftarrow\rangle = s|\uparrow\rangle - s|\downarrow\rangle$$

To "rotate" a spin state, apply rotation operator $R$:

$$(R \text{ operator depends on axis and angle of rotation.})$$

$$|\phi'\rangle = R|\phi\rangle$$

$$= R\left(a|\uparrow\rangle + b|\downarrow\rangle\right)$$
$$= a R|\uparrow\rangle + b R|\downarrow\rangle$$
Rotating by $\pi/2$

Rotate about the y-axis. We'd like:

$$R\left|\uparrow\right\rangle = \left|\to\right\rangle \quad R\left|\to\right\rangle = \left|\downarrow\right\rangle$$

$$R\left|\downarrow\right\rangle = \left|\leftarrow\right\rangle \quad R\left|\leftarrow\right\rangle = \left|\uparrow\right\rangle$$

But this is not possible!

Suppose

$$R\left|\uparrow\right\rangle = \left|\to\right\rangle = s\left|\uparrow\right\rangle + s\left|\downarrow\right\rangle$$

$$R\left|\downarrow\right\rangle = \left|\leftarrow\right\rangle = s\left|\uparrow\right\rangle - s\left|\downarrow\right\rangle$$

Then

$$R\left|\to\right\rangle = sR\left|\uparrow\right\rangle + sR\left|\downarrow\right\rangle$$

$$= (s^2 + s^2)\left|\uparrow\right\rangle + (s^2 - s^2)\left|\downarrow\right\rangle$$

$$= \left|\uparrow\right\rangle$$

$Uh$-$oh.$
Rotating by \( \pi/2 \)

How do we fix this? Try:

\[
R|\uparrow\rangle = |\rightarrow\rangle = s|\uparrow\rangle + s|\downarrow\rangle
\]

\[
R|\downarrow\rangle = \alpha|\leftarrow\rangle = \alpha(s|\uparrow\rangle - s|\downarrow\rangle)
\]

Then

\[
R|\rightarrow\rangle = sR|\uparrow\rangle + sR|\downarrow\rangle
\]

\[
= (s^2 + \alpha s^2)|\uparrow\rangle + (s^2 - \alpha s^2)|\downarrow\rangle
\]

\[
= |\downarrow\rangle \quad \text{(provided } \alpha = -1)\]

The actual rotation rule
Rotating by $2\pi$

A peculiar minus sign:

$$\text{Rot}(2\pi) = R \cdot R \cdot R \cdot R = R^4$$

$$|\phi\rangle \rightarrow R^4 |\phi\rangle = -|\phi\rangle$$

Rot($2\pi$) $\neq$ Rot(0), but

Rot($4\pi$) = Rot(0)

Spin-1/2 particles see a "4$\pi$" world!

The actual rotation rule:

$$R |\uparrow\rangle = |\rightarrow\rangle \quad R |\rightarrow\rangle = |\downarrow\rangle$$

$$R |\downarrow\rangle = -|\leftarrow\rangle \quad R |\leftarrow\rangle = |\uparrow\rangle$$
Rot(2\pi) \neq \text{Rot}(0) \text{ but } \text{Rot}(4\pi) = \text{Rot}(0)

First thought

This is totally weird. How can this be right??

Second thought

Maybe this is not so bad. Minus sign is unobservable!

\[ |\phi\rangle = a |\uparrow\rangle + b |\downarrow\rangle \]
\[ P(\uparrow) = |a|^2 = -|a|^2 \]
\[ P(\downarrow) = |b|^2 = -|b|^2 \]

Probabilities don't change if |\phi\rangle \rightarrow -|\phi\rangle

Third thought

On the other hand ....